

ON REDUCTIVE AUTOMORPHISM GROUPS OF REGULAR EMBEDDINGS

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ABSTRACT. Let G be a connected reductive complex algebraic group acting on a smooth complete complex algebraic variety X . We assume that X under the action of G is a *regular embedding*, a condition satisfied in particular by smooth toric varieties and flag varieties. For any set \mathcal{D} of G -stable prime divisors, we study the action on X of the group $\mathrm{Aut}^\circ(X, \mathcal{D})$, the connected automorphism group of X stabilizing \mathcal{D} . We determine a Levi subgroup A of $\mathrm{Aut}^\circ(X, \mathcal{D})$ and we compute relevant invariants of X as a spherical A -variety. As a byproduct, we obtain a description of the open A -orbit on X and the inclusion relation between A -orbit closures.

1. INTRODUCTION

In the 1970's Demazure described the connected automorphism groups of two distinguished classes of algebraic varieties equipped with the action of a connected reductive group G : the complete homogeneous spaces G/P for P a parabolic subgroup (see [De77]), and the smooth complete toric varieties, with G abelian (see [De70]). In the case of $X = G/P$, the group G goes surjectively onto the connected automorphism group $\mathrm{Aut}^\circ(X)$ except for three particular cases (with G a simple group) and products $(G_1 \times G_2)/(P_1 \times P_2)$ where $P_1 \subseteq G_1$, $P_2 \subseteq G_2$, and G_1/P_1 is one of these three exceptions. In the case where X is a toric G -variety, the image of G in $\mathrm{Aut}^\circ(X)$ is a maximal torus of the latter, and the corresponding root datum of $\mathrm{Aut}^\circ(X)$ is completely determined by the spaces of global sections $H^0(X, \mathcal{O}_X(Y))$, with Y varying in the set of G -stable prime divisors of X .

These two classes of G -varieties admit a common generalization: the *regular embeddings*, here also called G -regular embeddings or G -regular varieties, defined independently in [BDP90] and [Gi89]. With the additional assumption of completeness, Bien and Brion showed that these varieties correspond to a relevant class of spherical varieties, namely the smooth, complete, and *toroidal* ones (see [BB96]).

The spaces $H^0(X, \mathcal{O}_X(Y))$ play again an important role, especially for the case where X is *wonderful* in the sense of [Lu01] (see [Br07]), although they do not yield a direct description of $\mathrm{Aut}^\circ(X)$ if X is not toric. Also, the group $\mathrm{Aut}^\circ(X)$ may be non-reductive. Nevertheless, X is a

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spherical variety under the action of A , where A is any reductive subgroup $\text{Aut}^\circ(X)$ containing the image of G , therefore it is natural to study the relationship between invariants of X as a spherical G -variety, the structure of A , and invariants of X with respect to the A -action. The results of [AG10] are also related to this problem, and classify those toric varieties that are homogeneous under the action of a semisimple group.

In this paper we provide a complete description of the action of A on X if A is a Levi subgroup of $\text{Aut}^\circ(X, \mathcal{D})$. Here \mathcal{D} is any subset of the set ∂X of G -stable prime divisors of X , and $\text{Aut}^\circ(X, \mathcal{D})$ is the connected component of the group of automorphisms of X stabilizing each element of \mathcal{D} .

Our approach is based on the analysis of the following filtration:

$$\theta(G) \subseteq \text{Aut}^\circ(X, \partial X) \subseteq \text{Aut}^\circ(X, \mathcal{D} \cup (\partial X)^\ell) \subseteq \text{Aut}^\circ(X, \mathcal{D}),$$

where $\theta(G)$ is the image of G in $\text{Aut}^\circ(X)$ and $(\partial X)^\ell$ is a certain subset of ∂X (see Definition 2.5). The main motivation is the fact that the groups $\text{Aut}^\circ(X, \partial X)$ and $\text{Aut}^\circ(X, \mathcal{D} \cup (\partial X)^\ell)$ are reductive and X is regular under their actions, whereas both statements may fail for $\text{Aut}^\circ(X, \mathcal{D})$.

For the group $\text{Aut}^\circ(X, \partial X)$, we show in §4 that it is completely determined by results of [Br07] and [Pe09]. Then we consider $\mathcal{D}' = \mathcal{D} \cup (\partial X)^\ell$ and show in §7 that $\text{Aut}^\circ(X, \mathcal{D}')$ is reductive, and that it can be studied using a certain G -equivariant map $X \rightarrow \mathbb{X}$, where \mathbb{X} is a wonderful G -variety canonically associated with X . Namely, the group $\text{Aut}^\circ(X, \mathcal{D}')$ (up to a central torus) is obtained lifting to X the action of the universal cover of a certain semisimple subgroup of $\text{Aut}^\circ(\mathbb{X})$. The latter is known thanks to the results of [Pe09], which are somewhat similar to Demazure's theorem on flag varieties: the image of G is the whole $\text{Aut}^\circ(\mathbb{X})$, up to some exceptions that can be explicitly described.

It is worth noticing that \mathbb{X} is obtained from X using a procedure called *wonderful closure*, which is closely related to the well-known construction of the *spherical closure* of a spherical subgroup of G .

For the group $\text{Aut}^\circ(X, \mathcal{D})$, we show that it is enough to deal with the case where \mathcal{D} contains $(\partial X) \setminus (\partial X)^\ell$ (see the discussion at the end of §6). Under this assumption we show in §10 how to recover a Levi subgroup A of $\text{Aut}^\circ(X, \mathcal{D})$ by an analysis of the fibers of the map $X \rightarrow \mathbb{X}$, which are finite unions of toric varieties.

We also give an explicit combinatorial description of all the invariants commonly associated to X as a spherical A -variety, invariants which uniquely determine X up to A -equivariant isomorphisms thanks to the classification of spherical varieties.

In particular, we describe both the invariants associated to the open A -orbit on X , the so-called *Luna invariants*, and the invariants associated to X considered as an embedding of its open A -orbit, according to the Luna-Vust theory of embeddings of spherical homogeneous

spaces. Thanks to this theory, this accounts for a complete description of the structure of the A -orbits on X .

We also discuss explicitly in §8 and §9 the two special cases of G semisimple and G abelian.

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Notations. Through this paper G is a connected reductive linear algebraic group over the field of complex numbers \mathbb{C} . We assume that $G = G' \times C$ where C is an algebraic torus and G' is semisimple and simply connected. We denote by \mathbb{G}_m the multiplicative algebraic group of non-zero complex numbers.

We fix a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$. We denote by B^- the Borel subgroup of G such that $B \cap B^- = T$. If H is any algebraic group then we denote by $Z(H)$ its center, by H° its connected subgroup containing the unit element e_H , and by $\mathcal{X}(H)$ the set of its characters, i.e. algebraic group homomorphisms $H \rightarrow \mathbb{G}_m$. If V is an H -module, then we denote by $V^{(H)}$ the set of *non-zero* H -semiinvariants of V , and for any $\chi \in \mathcal{X}(H)$ we set

$$V_\chi^{(H)} = \{v \in V \setminus \{0\} \mid hv = \chi(h)v \ \forall h \in H\}.$$

If $H \subseteq K$ are subgroups of G , then we denote by $\pi^{H,K}: G/H \rightarrow G/K$ the natural map sending $gH \in G/H$ to $gK \in G/K$.

For any subset R of a \mathbb{Z} -module Λ , we denote by R^\vee (resp. R^\perp) the subset of $\text{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Q})$ of all elements that are ≥ 0 (resp. $= 0$) on R . We define in the same way *mutatis mutandis* the subsets $R^\vee, R^\perp \subseteq \Lambda$ for $R \subseteq \text{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Q})$.

The term *algebraic variety* (or simply *variety*) stands here for separated, reduced and irreducible scheme of finite type over the field \mathbb{C} , and all actions of algebraic groups on varieties are assumed to be algebraic. If X is a variety, the connected component containing id_X of its automorphism group is denoted by $\text{Aut}^\circ(X)$. If a connected algebraic group H acts on X , we denote by

$$\theta_{H,X}: H \rightarrow \text{Aut}^\circ(X)$$

the corresponding homomorphism.

If X is a G -variety, we denote by $\text{Pic}^G(X)$ the group of isomorphism classes of G -linearized invertible sheaves. If X is normal and Y is a Cartier divisor, then the invertible sheaf $\mathcal{O}_X(Y)$ admits a (non unique) G -linearization (see [KKLV89, Remark after Proposition 2.4]). If in addition X is complete and Y is a G -stable prime divisor, we will always assume that the G -linearization is chosen in such a way that the induced G -action on $H^0(X, \mathcal{O}_X(Y))$ is equal to the action inherited via the usual inclusion $H^0(X, \mathcal{O}_X(Y)) \subset \mathbb{C}(X)$.

2. COMPLETE REGULAR EMBEDDINGS

Definition 2.1. Suppose that an irreducible G -variety X has an open G -orbit. Then X is G -regular (or a G -regular embedding) if for any $x \in X$:

- (1) the closure \overline{Gx} of its orbit is smooth, and it is the transversal intersection of the G -stable prime divisors containing it;
- (2) the stabilizer G_x has a dense orbit on the normal space in X to the orbit Gx in the point x .

As an immediate consequence of the definition, a G -regular embedding is smooth and has only a finite number of G -orbits. Examples of G -regular embeddings are the G -homogeneous spaces for any G , and if G is an algebraic torus then any smooth toric G -variety. Other examples come from the family of *spherical varieties*, which are by definition irreducible normal G -varieties with a dense B -orbit.

More precisely, suppose that a G -variety X is smooth and complete. Then X is G -regular if and only if it is spherical and *toroidal*, i.e. any B -stable prime divisor containing a G -orbit is also G -stable (see [BB96, Proposition 2.2.1]).

We review some relevant invariants associated to any spherical G -variety X . They are actually invariants under birational G -equivariant maps, therefore they only depend on the open G -orbit of X . If x_0 is a point on this orbit, then we also denote the orbit Gx_0 simply by G/H , where $H = G_{x_0}$ is called a *generic stabilizer* of X . In this case, H is also called a *spherical subgroup*, and (X, x_0) (or simply X) is called an *embedding* of G/H . A *morphism* between two embeddings (X, x_0) and (X', x'_0) is a G -equivariant map $X \rightarrow X'$ sending x_0 to x'_0 .

We will always assume that x_0 is chosen in such a way that Bx_0 is dense in X . Then H is also called a *B -spherical subgroup*.

Definition 2.2. Let X be a spherical G -variety with open G -orbit G/H .

- (1) We define¹ the lattice

$$\Lambda_G(X) = \{ \chi \in \mathcal{X}(B) \mid \mathbb{C}(X)_\chi^{(B)} \neq \emptyset \},$$

whose rank is by definition the *rank* of X .

- (2) We define

$$N_G(X) = \text{Hom}_{\mathbb{Z}}(\Lambda_G(X), \mathbb{Q}).$$

- (3) We define $\Delta_G(X)$ to be the set of *colors* of X , i.e. the B -stable prime divisors of X having non-empty intersection with the open G -orbit G/H of X .

¹We ignore the dependence on B of all the invariants we define. This is justified by the fact that for any reductive group under consideration the choice of a Borel subgroup will be either unique (when the group is abelian) or always explicitly fixed.

- (4) For any discrete valuation $\nu: \mathbb{C}(X) \setminus \{0\} \rightarrow \mathbb{Q}$ we define an element $\rho_{G,X}(\nu) \in N_G(X)$ with the formula

$$\langle \rho_{G,X}(\nu), \chi \rangle = \nu(f_\chi),$$

where $f_\chi \in \mathbb{C}(X)_\chi^{(B)}$. If D is a prime divisor of X and ν_D is the associated discrete valuation, then we will also write $\rho_{G,X}(D)$ for $\rho_{G,X}(\nu_D)$.

- (5) We define

$$V_G(X) = \{ \rho_{G,X}(\nu) \mid \nu \text{ is } G\text{-invariant} \},$$

which is a polyhedral convex cone of maximal dimension in $N_G(X)$; we denote its linear part by $V_G^\ell(X)$.

- (6) We define the *boundary* of X , denoted by $\partial_G X$, to be the set of the irreducible components of $X \setminus (G/H)$.

For the above, and for all the invariants defined later, we will drop the indices G and X whenever it is clear which group and which variety are considered. In loose terms the colors of X can also be considered as invariants under G -equivariant birational maps, since they are the closures in X of the colors of G/H .

The Luna-Vust theory of embeddings of homogeneous spaces specializes for spherical toroidal varieties in the following way (for details and proofs see [Kn96]).

Definition 2.3. Let X be a G -regular embedding, and Y an irreducible G -stable locally closed subvariety. Then we define $c_{X,Y} \subseteq N(X)$ to be the polyhedral convex cone generated by $\rho(D_1), \dots, \rho(D_n)$, where D_1, \dots, D_n are the B -stable prime divisors containing Y . The *fan* of X is defined as

$$\mathcal{F}_G(X) = \{ c_{X,Y} \mid Y \text{ a } G\text{-orbit of } X \}.$$

Notice that since X is toroidal then the divisors D_1, \dots, D_n above are also G -stable for any Y . The collection of convex cones $\mathcal{F}(X)$ satisfies the following properties:

- (1) each cone of $\mathcal{F}(X)$ is contained in $V(G/H)$, it is strictly convex, and all its faces belong to $\mathcal{F}(X)$,
- (2) any element of $V(G/H)$ belongs to the relative interior of at most one cone of $\mathcal{F}(X)$.

The map $X \mapsto \mathcal{F}(X)$ induces a bijection between toroidal embeddings of G/H (up to isomorphism of embeddings) and *fans*, i.e. collections of strictly convex polyhedral convex cones satisfying (1) and (2).

The *support* of a fan \mathcal{F} is defined as

$$\text{supp } \mathcal{F} = \bigcup_{c \in \mathcal{F}} c.$$

The embedding X is complete if and only if $\text{supp } \mathcal{F}(X) = V(X)$, and it is smooth if and only if for each $c \in \mathcal{F}(X)$ there exists a basis $\gamma_1, \dots, \gamma_r$ of $\Lambda(X)$ and an integer k between 1 and r such that

$$c = \{\gamma_1, \dots, \gamma_k\}^\vee.$$

For later reference, we recall that if a spherical embedding X is not toroidal, then it is also described by a similar datum, called a fan of *colored convex cones*. Here, the convex cone associated to a G -orbit $Y \subseteq X$ is replaced by the pair $(c_{X,Y}, d_{X,Y})$ where $d_{X,Y}$ is the set of colors containing Y , and $c_{X,Y}$ is defined as above.

In general, the set $V(X)$ is also a polyhedral convex cone, of maximal dimension, and its linear part $V^\ell(X)$ has the same dimension (as a \mathbb{Q} -vector space) of $N_G H/H$ (as a complex algebraic group). The equations defining the maximal proper faces of $V(X)$ are linearly independent (see [Br90, Corollaire 3.3]). In other words, there always exist $\sigma_1, \dots, \sigma_k \in \Lambda(X)$ that are indivisible, linearly independent, and such that

$$V(X) = \{-\sigma_1, \dots, -\sigma_k\}^\vee.$$

Definition 2.4. The elements $\sigma_1, \dots, \sigma_k$ above are uniquely determined by G/H and called the *spherical roots* of X ; their set is denoted as

$$\Sigma_G(X) = \{\sigma_1, \dots, \sigma_k\}.$$

The map $Y \mapsto c_{X,Y}$ sends a G -orbit of codimension d in X to a cone of dimension d , and this restricts to a bijection between the boundary ∂X and the set of 1-dimensional cones in $\mathcal{F}(X)$.

Definition 2.5. For a subset $\mathcal{D} \subseteq \partial X$, we define the subsets

$$\mathcal{D}^\ell = \{Y \in \mathcal{D} \mid c_{X,Y} \subset V^\ell(X)\}$$

and

$$\mathcal{D}^{n\ell} = \mathcal{D} \setminus \mathcal{D}^\ell.$$

3. SPHERICAL AND WONDERFUL CLOSURE

In this section we recall the notion, introduced in [Lu01], of the spherical closure \overline{H} of a spherical subgroup $H \subseteq G$. We also define another subgroup containing H , called its wonderful closure. This is essentially already known, but not yet found in the literature. We gather at first some results from [Lu01, §6].

An element n of the normalizer N of H induces a G -equivariant isomorphism $G/H \rightarrow G/H$ given by $gH \mapsto gnH$. This induces an action of N on the set of colors $\Delta(G/H)$: the *spherical closure* \overline{H} of H is defined as the kernel of this action.

If $\overline{H} = H$ then we say that H is *spherically closed*, and for any spherical subgroup H the spherical closure \overline{H} is itself spherically closed. This is well known, but for lack of a detailed reference we provide a proof, also because $N_G \overline{H}$ may well be strictly bigger than $N_G H$.

Proposition 3.1. *For any spherical subgroup $H \subseteq G$, the spherical closure \overline{H} is spherically closed.*

Proof. Since \overline{H} is contained in $N_G H$ the quotient \overline{H}/H is diagonalizable (see [Kn94, Theorem 6.1]), and thus H is defined inside \overline{H} as intersection of kernels of some characters. The colors of G/\overline{H} generate $\text{Pic}^G(G/\overline{H})$ (see [Br89, Proposition 2.2]) and the latter is isomorphic to $\mathcal{X}(\overline{H})$ (see [KKV89, §3.1]), therefore \overline{H} acts trivially on $\mathcal{X}(\overline{H})$.

This implies that \overline{H} normalizes H . By definition, it fixes all colors of G/\overline{H} , but these correspond to the colors of G/H via the natural map $\pi^{H,\overline{H}}: G/H \rightarrow G/\overline{H}$. Hence $\overline{H} \subseteq \overline{H}$. \square

For later convenience we report the following auxiliary result. Recall that whenever $H \subseteq K$ are spherical subgroups of G , the lattice $\Lambda(G/H)$ is contained in the lattice $\Lambda(G/K)$, since B -semiinvariant functions can be lifted from G/H to G/K via the map $\pi^{H,K}: G/H \rightarrow G/K$. We sometimes denote this inclusion as a map $(\pi^{H,K})^*: \Lambda(G/H) \rightarrow \Lambda(G/K)$, which induces a surjection $\pi_*^{H,K}: N(G/H) \rightarrow N(G/K)$. Moreover, we have $\pi_*^{H,K}(V(G/H)) = V(G/K)$ and $\ker \pi_*^{H,\overline{H}} = V^\ell(G/H)$ (see [Kn96, Theorem 4.4 and Theorem 6.1]).

Lemma 3.2. *Let $H \subseteq K \subseteq \overline{H}$ be spherical subgroups. Then $(\pi_*^{H,K})^{-1}(V(G/H)) = V(G/K)$.*

Proof. The claim stems from $\pi_*^{H,K}(V(G/H)) = V(G/K)$, together with

$$\ker(\pi_*^{H,K}) \subseteq V^\ell(G/H).$$

This inclusion follows from the fact that $\pi_*^{K,\overline{H}} \circ \pi_*^{H,K} = \pi_*^{H,\overline{H}}$, and that the latter has kernel $V^\ell(G/H)$. \square

A class of subgroups slightly broader than the spherically closed ones is the following.

Definition 3.3. Suppose that $\Sigma(G/H)$ is a basis of $\Lambda(G/H)$. Then we say that H is a *wonderful* subgroup of G . In this case there exists a fan \mathcal{F} having only one maximal cone equal to $V(G/K)$; the associated toroidal embedding is denoted by $\mathbb{X}(G/H)$.

If H is wonderful then the embedding $\mathbb{X}(G/H)$ is smooth, has a unique closed G -orbit and it is *wonderful* in the sense of [Lu01]. A fundamental theorem of Knop (see [Kn96, Corollary 7.6]) states that a spherically closed subgroup is wonderful.

Example 3.4. The converse of the above statement is false: for example, if $G = \text{SO}(2n+1)$ with $n \geq 2$, then $H = \text{SO}(2n)$ is a wonderful subgroup, with $\overline{H} = N_{\text{SO}(2n+1)} \text{SO}(2n) \neq H$ (see [Wa96, cases 7B, 8B of Table 1]).

It is possible to define canonically a minimal wonderful subgroup \widehat{H} between H and \overline{H} . As a byproduct, the automorphism groups of regular embeddings of G/H are more directly related to the automorphism group of $\mathbb{X}(G/\widehat{H})$ than to that of $\mathbb{X}(G/\overline{H})$.

Definition 3.5. Let H and I be spherical subgroups of G . Then I is a *wonderful closure* of H if it is wonderful, satisfies $H \subseteq I \subseteq \overline{H}$, and is minimal with respect to these properties.

We will show that a wonderful closure always exists and is unique; for this we need to describe combinatorially all spherical subgroups having spherical closure equal to \overline{H} .

Let us fix a spherically closed subgroup K , and consider the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Lambda(G/K) & \xrightarrow{\overline{\rho}} & \mathrm{Pic}^G(\mathbb{X}(G/K)) & \xrightarrow{\tau} & \mathrm{Pic}^G(G/K) \longrightarrow 0 \\ & & & & \downarrow \sigma & & \\ & & & & \mathrm{Pic}^G(G/B) & & \end{array}$$

where the row is exact (see also [Br07, Proposition 2.2.1]).

The map τ is the pullback along the inclusion $G/K \rightarrow \mathbb{X}(G/K)$. For σ , observe that $\mathbb{X}(G/K)$ has a unique closed G -orbit Z , which is projective and therefore comes with a natural projection map $G/B \rightarrow Z$. The map σ is then the pullback along the composition $G/B \rightarrow Z \rightarrow \mathbb{X}(G/K)$.

The map $\overline{\rho}$ is defined in the following way: for any $\chi \in \Lambda(G/K)$ we take a function $f_\chi \in \mathbb{C}(G/K)_\chi^{(B)}$ and consider the G -stable part $D = \mathrm{div}(f_\chi)^G$ of $\mathrm{div}(f_\chi)$. Then we set $\overline{\rho}(\chi) = \mathcal{O}_{\mathbb{X}}(-D)$, which admits a unique G -linearization such that C acts trivially on the total space of the bundle.

These maps admit also a combinatorial definition, using the fact that $G = C \times G'$ and $K \supseteq C$, that $\Delta(G/K)$ is a basis of $\mathrm{Pic}(\mathbb{X}(G/K))$ (see [Br89, Proposition 2.2]), and the isomorphisms $\mathrm{Pic}^G(G/K) \cong \mathcal{X}(K)$, $\mathrm{Pic}^G(G/B) \cong \mathcal{X}(B)$. The resulting diagram

$$(3.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \Lambda(G/K) & \xrightarrow{\overline{\rho}} & \mathcal{X}(C) \times \mathbb{Z}^\Delta & \xrightarrow{\tau} & \mathcal{X}(K) \longrightarrow 0 \\ & & & & \downarrow \sigma & & \\ & & & & \mathcal{X}(B) & & \end{array}$$

where $\Delta = \Delta(G/K)$, is also described in details in [Lu01, §6.3]. The map $\overline{\rho}$ is defined as:

$$\overline{\rho}(\chi) = (\chi|_C, \langle \rho_{G,G/K}(\cdot), \chi \rangle),$$

and $\sigma \circ \overline{\rho}$ is the identity on $\Lambda(G/K)$ (see loc.cit.).

Lemma 3.6. [Lu01, Lemme 6.3.1, Lemme 6.3.3] *Let $K \subseteq G$ be a spherically closed subgroup. The application*

$$H \rightarrow \tau^{-1}(\mathcal{X}(K)^H)$$

is an inclusion-reversing bijection between the set of normal subgroups H of K such that K/H is diagonalizable, and the set of subgroups of $\mathcal{X}(C) \times \mathbb{Z}^\Delta$ containing $\bar{\rho}(\Lambda(G/K))$. If the restriction of σ to $\tau^{-1}(\mathcal{X}(K)^H)$ is injective then H is spherical.

Lemma 3.7. *For any spherical subgroup $H \subseteq G$ contained and normal in K and all $D \in \Delta(G/H)$ we have $\pi^{H,K}(D) \in \Delta(G/K)$, and*

$$\pi_*^{H,K}(\rho_{G,G/H}(D)) = \rho_{G,G/K}(\pi^{H,K}(D)).$$

Proof. Since H is normal in K , then K stabilizes the open set $BH \subseteq G$ acting by right multiplication on G (see also [BP87, First part of the proof of Proposition 5.1]). The complement $G \setminus BH$ is the union of $\pi^{\{e_G\},H}(E)$ for E varying in $\Delta(G/H)$, whence the first statement.

For the second statement, it is enough to show that a local equation of D on G/H can be chosen to be the pull-back of a function on G/K along $\pi^{H,K}$. Let E_1, \dots, E_n be all the distinct B -stable prime divisors of G such that $\pi^{\{e_G\},K}(E_i) = \pi^{H,K}(D)$. Since G is factorial we can choose a global equation $f_i \in \mathbb{C}[G]$ for each E_i , and consider the product $f = f_1 \cdot \dots \cdot f_n$.

The divisor $\text{div}(f)$ on G is B -stable under the left translation action of G on itself, but none of its components is G -stable therefore there exists an element $g \in G$ such that the function $f_0: x \mapsto f(gx)$ doesn't vanish on any divisor E_i . On the other hand $\text{div}(f)$ is K -stable under the right translation action of G on itself, thus f is K -semiinvariant under this action. The function f_0 is then also K -semiinvariant, with same K -eigenvalue. It follows that

$$F = \frac{f}{f_0}$$

is K -invariant with respect to the right translation action. In other words $F = (\pi^{\{e_G\},K})^*(\tilde{F})$ for some $\tilde{F} \in \mathbb{C}(G/K)$.

Now for some i_0 the divisor E_{i_0} satisfies $\pi^{\{e_G\},H}(E_{i_0}) = D$. The function F is equal to the pull-back of $(\pi^{H,K})^*(\tilde{F})$ along $\pi^{\{e_G\},H}$ and is a local equation of E_{i_0} on G , hence $(\pi^{H,K})^*(\tilde{F})$ is a local equation of $\pi^{\{e_G\},H}(E_{i_0}) = D$ on G/H : the lemma follows. \square

Thanks to Lemma 3.7, we can extend the map $\bar{\rho}$ to $\Lambda(G/H)$ in the following way.

Definition 3.8. We denote again by $\bar{\rho}$ the extension of the above map $\bar{\rho}: \Lambda(G/K) \rightarrow \mathcal{X}(C) \times \mathbb{Z}^\Delta$ to $\Lambda(G/H)$ given by the following formula:

$$\bar{\rho}(\chi) = (\chi|_C, \langle \rho_{G,G/H}(\cdot), \chi \rangle).$$

Lemma 3.9. *For any spherical subgroup $H \subseteq G$ contained and normal in K we have*

$$(3.2) \quad \bar{\rho}(\Lambda_G(G/H)) = \tau^{-1}(\mathcal{X}(K)^H).$$

Proof. The equality stems from the description of the map τ given in [Lu01, §6.3], see in particular [Lu01, Proof of Proposition 6.3]. Indeed, for any $\chi \in \Lambda(G/H)$ the image $\tau(\bar{\rho}(\chi))$ is the K -eigenvalue of a rational function f on G such that f is the pull-back of a rational function on G/H . Hence its K -eigenvalue is trivial on H .

For the other inclusion, Lemma 3.7 implies that

$$\bigcup_{D \in \Delta(G/H)} (\pi^{\{e_G\}, H})^{-1}(D) = \bigcup_{D \in \Delta(G/K)} (\pi^{\{e_G\}, K})^{-1}(D)$$

is the union of all prime divisors of G that are B -stable under left translation and H -stable (or equivalently K -stable) under right translation. As a consequence, if $\tau(\gamma, (n_D)_{D \in \Delta})$ is a K -character that is trivial on H , then $\chi = \sigma(\tau(\gamma, (n_D)_{D \in \Delta}))$ is the B -eigenvalue of a B -semiinvariant rational function on G/H . Since $\bar{\rho}(\chi) = (\gamma, (n_D)_{D \in \Delta})$, the proof is complete. \square

Proposition 3.10. *Let $H \subseteq G$ be a spherical subgroup, set $K = \overline{H}$ and $\Xi = \text{span}_{\mathbb{Z}} \Sigma(G/H)$. Then*

$$\Lambda(G/K) \subseteq \Xi \subseteq \Lambda(G/H).$$

The normal subgroup $\widehat{H} \subseteq K$ associated to $\bar{\rho}(\Xi)$ via the map of Lemma 3.6 is the unique wonderful closure of H . It has the same dimension of \overline{H} , and it is the unique wonderful subgroup between H and \overline{H} that satisfies $\Sigma_G(G/H) = \Sigma(G/\widehat{H})$. Moreover, the spherical closure of \widehat{H} is \overline{H} .

Proof. The inclusion $\Xi \subseteq \Lambda(G/H)$ is obvious. The map $\pi_*^{H,K}: N(G/H) \rightarrow N(G/K)$ has kernel $V^\ell(G/H)$, and satisfies $\pi_*(V(G/H)) = V(G/K)$. The other inclusion $\Lambda(G/K) \subseteq \Xi$ follows. Hence the subgroup \widehat{H} contains H .

The lattice $\Lambda(G/\widehat{H}) = \Xi$ has basis $\Sigma(G/H)$ since the spherical roots are always linearly independent. Since $\Lambda(G/\widehat{H})$ has finite index inside $V^\ell(G/H)^\perp$ and $\pi^{H,\widehat{H}}(V(G/H)) = V(G/\widehat{H})$ we deduce that $\Sigma(G/H) = \Sigma(G/\widehat{H})$.

If \widetilde{H} is another wonderful subgroup such that $H \subseteq \widetilde{H} \subseteq \overline{H}$, then $\Lambda(G/\widetilde{H})$ has also finite index in $V^\ell(G/H)^\perp$, and $\Sigma(G/\widetilde{H})$ is equal to $\Sigma(G/H)$ up to taking (positive) multiples of the elements of the latter. The dimension, minimality and uniqueness properties of \widehat{H} follow, since $\Lambda(G/\widetilde{H}) \subseteq \Lambda(G/\widehat{H})$ implies $\widetilde{H} \supseteq \widehat{H}$.

The last assertion follows from the last assertion of [Lu01, Lemme 6.3.3]: indeed the lattice $\sigma(\Phi')$ of loc.cit. is denoted here by Ξ , and the set S° of loc.cit. is here a subset of $\frac{1}{2}\Sigma(G/H)$. \square

4. AUTOMORPHISMS STABILIZING ALL G -ORBITS

From now on, X denotes a complete G -regular variety, with open G -orbit G/H .

Definition 4.1. For any subset $\mathcal{D} \subseteq \partial_G X$ of G -stable prime divisors we define

$$\text{Aut}^\circ(X, \mathcal{D}) = \{\phi \in \text{Aut}^\circ(X) \mid \phi(D) = D, \forall D \in \mathcal{D}\}.$$

Since X is G -regular, the group $\mathrm{Aut}^\circ(X, \partial_G X)$ is also the connected group of automorphisms of X stabilizing each G -orbit.

We recall now some results from [BB96] (see also [Br07]). The group $\mathrm{Aut}^\circ(X)$ is a linear algebraic group, with Lie algebra

$$\mathrm{Lie} \, \mathrm{Aut}^\circ(X) = H^0(X, \mathcal{T}_X)$$

where \mathcal{T}_X is the sheaf of sections of the tangent bundle of X . The structure of G -module on $\mathrm{Lie} \, \mathrm{Aut}^\circ(X)$, induced by the adjoint action of $\theta_{G,X}(G) \subseteq \mathrm{Aut}^\circ(X)$, is given in [BB96, Proposition 4.1.1] in terms of global sections of the line bundles $\mathcal{O}_X(D)$ where $D \in \partial_G X$.

Namely, there exists an exact sequence of G -modules

$$(4.1) \quad 0 \rightarrow \mathrm{Lie} \, \mathrm{Aut}^\circ(X, \partial_G X) \rightarrow \mathrm{Lie} \, \mathrm{Aut}^\circ(X) \rightarrow \bigoplus_{D \in \partial_G X} \frac{H^0(X, \mathcal{O}_X(D))}{\mathbb{C}} \rightarrow 0.$$

Moreover, for any $\mathcal{D} \in \partial_G X$ the Lie algebra of the subgroup $\mathrm{Aut}^\circ(X, \mathcal{D})$ is the inverse image of the sum

$$\bigoplus_{D \in (\partial_G X) \setminus \mathcal{D}} \frac{H^0(X, \mathcal{O}_X(D))}{\mathbb{C}}.$$

Definition 4.2. Let $0 \neq \gamma \in \mathcal{X}(B)$. If it exists, we denote by $X(\gamma)$ the uniquely determined element of $\partial_G X$ such that $H^0(X, \mathcal{O}_X(X(\gamma)))_\gamma^{(B)} \neq \emptyset$.

A particular case of $\mathrm{Aut}^\circ(X)$ has been studied in [Pe09], where X is a wonderful variety. Recall that C acts trivially on any wonderful G -variety, hence we can consider G' -varieties without loss of generality. Moreover, in this case $\mathrm{Aut}^\circ(X, \mathcal{D})$ is always semisimple and X is wonderful under its action (see [Br07, Theorem 2.4.2]). It is possible to summarize the results of [Pe09] as follows.

Theorem 4.3. [Pe09] *Let \mathbb{X} be a wonderful G' -variety and $\mathcal{D} \subseteq \partial_{G'} \mathbb{X}$. Decompose G' and \mathbb{X} into products*

$$G' = G'_1 \times \dots \times G'_n, \quad \mathbb{X} = \mathbb{X}_1 \times \dots \times \mathbb{X}_n,$$

with a maximal number of factors in such a way that G'_i acts non-trivially only on \mathbb{X}_i for all $i = 1, \dots, n$. Then

$$\mathrm{Aut}^\circ(\mathbb{X}, \mathcal{D}) = \mathrm{Aut}^\circ(\mathbb{X}_1, \mathcal{D}_1) \times \dots \times \mathrm{Aut}^\circ(\mathbb{X}_n, \mathcal{D}_n),$$

where $\mathcal{D}_i = \{D \cap \mathbb{X}_i \mid D \in \mathcal{D}\} \subseteq \partial_{G'_i} \mathbb{X}_i$. Moreover, if the image of G'_i in $\mathrm{Aut}^\circ(\mathbb{X}_i, \mathcal{D}_i)$ is a proper subgroup, then (G'_i, \mathbb{X}_i) appears in the lists of “exceptions” of [Pe09, §§3.2 – 3.6]. If $\mathcal{D} = \partial_{G'} \mathbb{X}$ then all such exceptional factors have rank 0 or 1.

Let now $\mathbb{X} = \mathbb{X}(G/\overline{H})$: the group $\text{Aut}^\circ(X, \partial_G X)$ is easily recovered from $\text{Aut}^\circ(\mathbb{X}, \partial_G \mathbb{X})$. Indeed, thanks to [Br07, Theorem 4.4.1], there exists a split exact sequence of Lie algebras

$$(4.2) \quad 0 \rightarrow \frac{\text{Lie } \overline{H}}{\text{Lie } H} \rightarrow \text{Lie Aut}^\circ(X, \partial_G X) \rightarrow \text{Lie Aut}^\circ(\mathbb{X}, \partial_G \mathbb{X}) \rightarrow 0.$$

It follows that $\text{Aut}^\circ(X, \partial_G X)$ is reductive, its connected center is $(\overline{H}/H)^\circ = (\widehat{H}/H)^\circ$, and its semisimple part can be computed using Theorem 4.3 and the lists of [Pe09].

We point out that in the above exact sequence we may as well use the variety $\mathbb{X} = \mathbb{X}(G/\widehat{H})$. Indeed, the results of [Br07, §4.4] hold (with same proofs) if we replace the spherical closure of H with its wonderful closure.

5. RELATING $\text{Aut}^\circ(X)$ TO $\text{Aut}^\circ(\mathbb{X})$

From now on, $\mathbb{X} = \mathbb{X}(G/\widehat{H})$ denotes the wonderful embedding of G/\widehat{H} . As a consequence of the last section, we may suppose from now on that $\theta_{G,X}(G) = \text{Aut}^\circ(X, \partial_G X)$ and that $\theta_{G,\mathbb{X}}(G) = \text{Aut}^\circ(\mathbb{X}, \partial_G \mathbb{X})$. Indeed, if this is not the case we may first apply Theorem 4.3 to \mathbb{X} , replace G'_i with the universal cover of $\text{Aut}^\circ(\mathbb{X}_i, \partial_{G_i} \mathbb{X}_i)$ for all i such that these groups are different, and then replace C with $C \times (\widehat{H}/H)^\circ$.

Thanks to [Kn96, Theorem 4.1], the natural surjection $\pi^{H,\widehat{H}}: G/H \rightarrow G/\widehat{H}$ extends to a surjective G -equivariant map

$$\pi: X \rightarrow \mathbb{X} = \mathbb{X}(G/\widehat{H}).$$

Definition 5.1. We denote by

$$X \xrightarrow{\psi} X' \xrightarrow{f} \mathbb{X}$$

the Stein factorization of the map $\pi: X \rightarrow \mathbb{X}$.

In [Br07, §4.4] it is shown that $\text{Aut}^\circ(X)$ acts on X' in such a way that ψ is equivariant; we denote the corresponding homomorphism as follows:

$$\psi_*: \text{Aut}^\circ(X) \rightarrow \text{Aut}^\circ(X').$$

Its kernel is the subgroup of automorphisms of X stabilizing each fiber of ψ .

Proposition 5.2. *The inclusions $Z(\theta_{G,X}(G))^\circ \subseteq \ker \psi_* \cap \theta_{G,X}(G) \subseteq Z(\theta_{G,X}(G))$ between subgroups of $\text{Aut}^\circ(X)$ hold. Moreover, there is a local isomorphism*

$$(5.1) \quad \text{Aut}^\circ\left(X, (\partial_G X)^{n_\ell}\right) \cong \theta_{G,X}(G') \ltimes (\ker \psi_*)^\circ$$

induced by the inclusion of both factors of the right hand side in $\text{Aut}^\circ(X)$.

Proof. The first inclusion stems from the fact that $C = Z(G)^\circ$ acts trivially on \mathbb{X} , hence also on X' . On the other hand, if $g \in G$ stabilizes all fibers of ψ , then it acts trivially on X' and also on \mathbb{X} . Therefore, to show the second inclusion, we only have to check that no simple factor of G acts trivially on \mathbb{X} but not on X . This is true because \widehat{H}/H is abelian.

Let us prove the last statement. Both groups on the right hand side of (5.1) are subgroups of $\text{Aut}^\circ(X, (\partial_G X)^{n_\ell})$: this is obvious for $\theta_{G,X}(G')$, so we only have to check it for $(\ker \psi_*)^\circ$. Notice that ψ maps any element D of $(\partial_G X)^{n_\ell}$ onto a proper G -stable closed subset of X' . It follows that D is an irreducible component of $\psi^{-1}(\psi(D))$, hence it is stable under the action of $(\ker \psi_*)^\circ$. It also follows that ψ_* maps $\text{Aut}^\circ(X, (\partial_G X)^{n_\ell})$ into $\text{Aut}^\circ(X', \partial_G X')$.

The intersection $(\ker \psi_*)^\circ \cap \theta_{G,X}(G')$ is finite thanks to the first part of the proof, and $(\ker \psi_*)^\circ$ is a normal subgroup of $\text{Aut}^\circ(X, (\partial_G X)^{n_\ell})$. It only remains to prove that $\text{Aut}^\circ(X, (\partial_G X)^{n_\ell})$ is generated by $\theta_{G,X}(G')$ and $(\ker \psi_*)^\circ$.

By [Br07, Theorem 4.4.1], we know that $\text{Aut}^\circ(X', \partial_G X')$ and $\text{Aut}^\circ(\mathbb{X}, \partial_G \mathbb{X})$ are both semisimple and locally isomorphic. It follows that the universal cover of $\text{Aut}^\circ(X', \partial_G X')$ acts on \mathbb{X} in such a way that f is equivariant. On the other hand no element of this universal cover could act trivially on X' and non-trivially on \mathbb{X} , hence $\text{Aut}^\circ(X', \partial_G X')$ itself acts on \mathbb{X} , preserving all G -orbits. This produces a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\theta_{G,X}} & \text{Aut}^\circ(X, (\partial_G X)^{n_\ell}) \\ \downarrow \theta_{G,\mathbb{X}} & & \downarrow \psi_* \\ \text{Aut}^\circ(\mathbb{X}, \partial_G \mathbb{X}) & \xleftarrow{f_*} & \text{Aut}^\circ(X', \partial_G X') \end{array}$$

where $\theta_{G,\mathbb{X}}$ is surjective by our assumptions. Therefore $\text{Aut}^\circ(X, (\partial_G X)^{n_\ell})$ is generated by $\theta_{G,X}(G)$ and $\ker(f_* \circ \psi_*)$. Notice that f_* has finite kernel, that the kernel of ψ_* contains $\theta_{G,X}(C)$, and that $\text{Aut}^\circ(X, (\partial_G X)^{n_\ell})$ is connected: we deduce that $\text{Aut}^\circ(X, (\partial_G X)^{n_\ell})$ is indeed generated by $\theta_{G,X}(G')$ and $(\ker \psi_*)^\circ$, and the proof is complete. \square

If we denote by

$$d\psi_*: \text{Lie Aut}^\circ(X) \rightarrow \text{Lie Aut}^\circ(X')$$

the corresponding homomorphism of Lie algebras, then the following corollary is an immediate consequence of the above proposition.

Corollary 5.3. *The subspace $\ker d\psi_* \subseteq \text{Lie Aut}^\circ(X)$ is G -stable, and its intersection with $\text{Lie } \theta_{G,X}(G)$ is equal to $\text{Lie } \theta_{G,X}(C)$. There exists a G -equivariant splitting of the exact sequence (4.1) such that*

$$\ker d\psi_* = \text{Lie } \theta_{G,X}(C) \oplus \bigoplus_{D \in (\partial_G X)^\ell} \frac{H^0(X, \mathcal{O}_X(D))}{\mathbb{C}}.$$

6. RESTRICTING AUTOMORPHISMS OF X TO FIBERS OF ψ

We study now the automorphisms of a generic fiber of ψ induced by automorphisms of X belonging to $\ker \psi_*$. For this it is convenient to exploit the *local structure* of spherical varieties.

Theorem 6.1. [Kn94, Theorem 2.3 and Proposition 2.4] *Let Y be a spherical G -variety. Let $P_{G,Y} \supseteq B$ be the stabilizer in G of the open B -orbit of Y , let $L_{G,Y}$ be the Levi subgroup of $P_{G,Y}$ containing T , and consider the following open subset of Y :*

$$Y_0 = Y \setminus \bigcup_{D \in \Delta_G(Y)} D.$$

Then there exists a closed $L_{G,Y}$ -stable and $L_{G,Y}$ -spherical subvariety $\mathcal{Z}_{G,Y}$ of Y_0 such that the map

$$\begin{aligned} P_{G,Y}^u \times \mathcal{Z}_{G,Y} &\rightarrow Y_0 \\ (p, z) &\mapsto pz \end{aligned}$$

is a $P_{G,Y}$ -equivariant isomorphism, where $L_{G,Y}$ acts on $P_{G,Y}^u \times \mathcal{Z}_{G,Y}$ by $l \cdot (p, z) = (lpl^{-1}, lz)$. The commutator subgroup $(L_{G,Y}, L_{G,Y})$ acts trivially on $\mathcal{Z}_{G,Y}$, and if Y is toroidal then every G -orbit meets $\mathcal{Z}_{G,Y}$ in an $L_{G,Y}$ -orbit.

Definition 6.2. We define $T_{G,Y}$ to be the quotient of $L_{G,Y}^r$ by the kernel of its action on $\mathcal{Z}_{G,Y}$.

We get back to our complete G -regular variety X . The torus $T_{G,X}$ is a subquotient of T , and $\mathcal{Z}_{G,X}$ is a spherical (toric) $T_{G,X}$ -variety, with lattice $\Lambda_{T_{G,X}}(\mathcal{Z}_{G,X}) = \mathcal{X}(T_{G,X}) = \Lambda_G(G/H)$ and fan of convex cones equal to $\mathcal{F}_G(X)$.

Definition 6.3. For any x' in the open G -orbit of X' we denote by $\kappa_{x'}$ the restriction map

$$\kappa_{x'}: (\ker \psi_*)^\circ \rightarrow \text{Aut}^\circ(X_{x'})$$

where $X_{x'} = \psi^{-1}(x')$.

Recall that H is chosen in such a way that BH is open in G , and $x_0 = eH \in G/H \subseteq X$. Let us consider $x'_0 = \psi(x_0)$: the fiber $X_{x'_0}$ is smooth and complete, and it is a toric variety under the action of the torus $S = (\widehat{H}/H)^\circ = H'/H$, where H' is the stabilizer of x'_0 .

Moreover, S acts naturally on G/H by G -equivariant automorphisms, and since S is connected this S -action extends to X , stabilizing all colors of X and all fibers of ψ . We may fix $\mathcal{Z}_{G,X'} \subset X'$ containing x'_0 , and choose $\mathcal{Z}_{G,X}$ so that

$$\mathcal{Z}_{G,X} = \psi^{-1}(\mathcal{Z}_{G,X'}) \cap X_0,$$

which implies that $\mathcal{Z}_{G,X}$ contains x_0 and is stable under the action of S .

The same action of S on $\mathcal{Z}_{G,X}$ can be realized sending S injectively into $T_{G,X}$, and then letting it act on $\mathcal{Z}_{G,X}$ via the restriction of the usual action of G on X . Indeed, if $nH \in S$

and $f \in \mathbb{C}(G/H)_\chi^{(B)}$, then $gH \mapsto f(gnH)$ also belongs to $\mathbb{C}(G/H)_\chi^{(B)}$, therefore there is a homomorphism (depending only on χ) $\iota_\chi: S \rightarrow \mathbb{C}^*$ such that $f(gnH) = \iota_\chi(n^{-1}H)f(gH)$ for all $g \in G$. This induces a homomorphism

$$\iota: S \rightarrow \text{Hom}(\Lambda_G(G/H), \mathbb{C}^*) \cong T_{G,X},$$

which can be shown to be injective, with image equal to the subtorus of $T_{G,X}$ corresponding to the subspace $V_G^\ell(G/H) \subseteq N_G(G/H)$ (see [Br97, Proof of Theorem 4.3]). Let us check that restricting to $\iota(S)$ the usual $T_{G,X}$ -action on $\mathcal{Z}_{G,X}$ yields the action described above. The intersection $\mathcal{Z}_{G,X} \cap G/H$ is dense in $\mathcal{Z}_{G,X}$, and $\mathcal{Z}_{G,X}$ is a toric $T_{G,X}$ -variety with lattice equal to $\Lambda_G(G/H)$: it follows that $\iota(nH)gH = gnH$, because

$$f(\iota(nH)gH) = \chi(\iota(n^{-1}H))f(gH) = \iota_\chi(n^{-1}H)f(gH) = f(gnH)$$

for all $nH \in S$, $gH \in \mathcal{Z}_{G,X} \cap G/H$, $\chi \in \Lambda_G(G/H)$ and $f \in \mathbb{C}(G/H)_\chi^{(B)}$.

The fiber $X_{x'_0}$ is also the fiber over x'_0 of the S -equivariant map $\mathcal{Z}_{G,X} \rightarrow \mathcal{Z}_{G,X'}$, which implies that its fan of convex cones is

$$(6.1) \quad \mathcal{F}_S(X_{x'_0}) = \{c \mid c \in \mathcal{F}_G(X), c \subset V_G^\ell(G/H)\}.$$

Since the S -boundary of $X_{x'_0}$ is given intersecting $X_{x'_0}$ with the elements of $(\partial_G X)^\ell$, there is an exact sequence of S -modules

$$(6.2) \quad 0 \rightarrow \text{Lie } S \rightarrow \text{Lie Aut}^\circ(X_{x'_0}) \rightarrow \bigoplus_{D \in (\partial_G X)^\ell} \frac{H^0(X_{x'_0}, \mathcal{O}_X(D \cap X_{x'_0}))}{\mathbb{C}} \rightarrow 0.$$

Lemma 6.4. *If $V \subseteq \ker d\psi_*$ is a simple G -submodule and x' is in the open B -orbit of X' , then $d\kappa_{x'}(V) = d\kappa_{x'}(\mathbb{C}v)$, where $v \in V$ is a highest weight vector.*

Proof. We may assume that $x' = x'_0$ and that $v = [s] \in H^0(X, \mathcal{O}_X(D))/\mathbb{C}$ for some $D \in (\partial X)^\ell$, in view of Corollary 5.3. From the expression in local coordinates of [BB96, Remark after Proposition 2.3.2] and the proof of [BB96, Proposition 4.1.1], we see that $d\kappa_{x'_0}(v)$ is sent by the surjective map of (6.2) to $[s|_{X_{x'_0}}]$ where $s|_{X_{x'_0}}$ is a section of $\mathcal{O}_{X_{x'_0}}(D \cap X_{x'})$.

If s is a B -eigenvector then its zeros are B -stable. On the other hand, since Bx'_0 is open in X' , the only zeros of s intersecting $X_{x'_0}$ are G -stable. It also follows that $(gs)|_{X_{x'_0}}$ and $s|_{X_{x'_0}}$ have the same zeros (hence are linearly dependent) for any $g \in G$ such that gx_0 doesn't lie on any color of G/H . This is true for g lying in a dense subset U of G , and since V is generated as a vector space by elements of the form $[gs]$ for $g \in U$, the lemma follows. \square

Lemma 6.5. *Let $i = 1, 2$ and $0 \neq \gamma_i \in \Lambda_G(X)$ be such that $X(\gamma_i)$ exists, with $X(\gamma_i) \in (\partial X)^\ell$. Suppose that $\langle m, \gamma_1 \rangle = -\langle m, \gamma_2 \rangle$ for all $m \in V_G^\ell(X)$. Then*

$$\langle \rho_{G,X}(D), \gamma_i \rangle = 0$$

for all $i = 1, 2$, for all $D \in (\partial_G X)^{n\ell}$ and for all $D \in \Delta_G(X)$.

Proof. Consider the wonderful variety \mathbb{X} . Both sets $\rho_{G,\mathbb{X}}(\Delta_G(\mathbb{X}))$ and $\rho_{G,\mathbb{X}}(\partial_G\mathbb{X})$ generate $N_G(\mathbb{X})$ as a vector space, and the convex cone generated by $\rho_{G,\mathbb{X}}(\Delta_G(\mathbb{X}))$ contains $-\rho_{G,\mathbb{X}}(\partial_G\mathbb{X})$ (see [Br07, Lemma 2.1.2]). On the other hand, the set $\pi_*(\rho_{G,X}((\partial_G X)^{n_\ell})) \subset N_G(\mathbb{X})$ generates the same convex cone $V_G(\mathbb{X})$ generated by $\rho_{G,\mathbb{X}}(\partial_G\mathbb{X})$, and $\pi_*(\rho_{G,X}(\Delta_G(X))) = \rho_{G,\mathbb{X}}(\Delta_G(\mathbb{X}))$. It follows that there exists a linear combination

$$v = \sum_{Y \in (\partial_G X)^{n_\ell}} n_Y \rho_{G,X}(Y) + \sum_{Z \in \Delta_G(X)} n_Z \rho_{G,X}(Z) \in V_G^\ell(X)$$

where all the coefficients n_Y and n_Z are positive. From the assumptions on the characters γ_i , all the elements $\rho_{G,X}(Y)$ and $\rho_{G,X}(Z)$ above are non-negative on both γ_1 and γ_2 : we deduce that $\langle v, \gamma_i \rangle \geq 0$, which yields $\langle v, \gamma_i \rangle = 0$. The lemma follows. \square

In the next sections we will investigate $\text{Aut}^\circ(X, \mathcal{D})$ for any subset $\mathcal{D} \subseteq \partial_G X$, using the results above. It is harmless to assume that each $E \in \mathcal{E} = \partial_G X \setminus \mathcal{D}$ is not stable under $\text{Aut}^\circ(X, \mathcal{D})$, and it is convenient to treat separately the two subsets $\mathcal{E}^{n_\ell}, \mathcal{E}^\ell$ of \mathcal{E} .

More precisely, we first consider in §7 the special case where $\mathcal{E}^\ell = \emptyset$, i.e. $\mathcal{D} \supseteq (\partial_G X)^\ell$. We determine the group $\text{Aut}^\circ(X, \mathcal{D})$: it is obtained lifting from \mathbb{X} to X the action of a certain subgroup of $\text{Aut}^\circ(\mathbb{X})$, it is reductive and under its action X is G -regular, with boundary \mathcal{D} . Finally, we compute the related fan of convex cones.

For a general \mathcal{D} , we apply the above results to X where G is replaced by $\tilde{G} = \text{Aut}^\circ(X, \mathcal{D} \cup \mathcal{E}^\ell)$. It turns out (see Corollary 7.19) that the elements of \mathcal{E}^ℓ lie on the linear part of the valuation cone both with respect to the G -action and to the \tilde{G} -action.

Therefore we may finally replace G with the group \tilde{G} , and develop further analysis on $\text{Aut}^\circ(X, \mathcal{D})$ where now X is a complete \tilde{G} -regular variety satisfying $\mathcal{D} \supseteq (\partial_{\tilde{G}} X)^{n_\ell}$. This will be done in §10, after discussing the special cases of G abelian (§8) and G semisimple (§9).

7. G -STABLE PRIME DIVISORS NOT ON THE LINEAR PART OF THE VALUATION CONE

In this section we study $\text{Aut}^\circ(X, \mathcal{D})$ under the assumption that $\mathcal{D} \supseteq (\partial_G X)^\ell$. We also suppose that \mathcal{D} contains all the G -stable prime divisors D that satisfy $H^0(X, \mathcal{O}_X(D)) = \mathbb{C}$, since these prime divisors do not move under the action of $\text{Aut}^\circ(X)$ anyway.

Before stating the main result of this section, Theorem 7.8, we need to establish a correspondence between the divisors in $\partial_G X \setminus \mathcal{D}$ and certain boundary divisors of \mathbb{X} .

Recall that since \mathbb{X} is wonderful the set $-\rho_{G,\mathbb{X}}(\partial_X \mathbb{X})$ is a basis of $N_G(\mathbb{X})$, dual to $\Sigma_G(\mathbb{X})$.

Definition 7.1. For an element $D \in \partial_G \mathbb{X}$, we denote by σ_D the spherical root of \mathbb{X} dual to $-\rho_{G,\mathbb{X}}(D)$.

Since $\Lambda_G(\mathbb{X})$ is a sublattice of $\Lambda_G(X)$, we consider σ_D also as an element of the latter.

Also recall that, thanks to [Br07, Theorem 2.2.3], if $D \in \partial_G \mathbb{X}$ satisfies $H^0(\mathbb{X}, \mathcal{O}_{\mathbb{X}}(D)) \neq \mathbb{C}$ then $H^0(\mathbb{X}, \mathcal{O}_{\mathbb{X}}(D))/\mathbb{C}$ is irreducible with highest weight σ_D .

Lemma 7.2. *Let $E \in \mathcal{E} = \partial_G X \setminus \mathcal{D}$. Then:*

- (1) *the image $\pi(E)$ is an element of $\partial_G \mathbb{X}$, with $H^0(\mathbb{X}, \mathcal{O}_{\mathbb{X}}(\pi(E))) \neq \mathbb{C}$, and E is the only element of $\partial_G X$ whose image is $\pi(E)$;*
- (2) *we have*

$$\pi_*(\rho_{G,X}(E)) = \rho_{G,\mathbb{X}}(\pi(E)),$$

and

$$(7.1) \quad \forall c \in \mathcal{F}_G(X) \setminus \{c_{X,E}\}, c \text{ 1-dimensional: } c \subset \sigma_{\pi(E)}^\perp;$$

- (3) *the G -modules $H^0(X, \mathcal{O}_X(E))$ and $H^0(\mathbb{X}, \mathcal{O}_{\mathbb{X}}(\pi(E)))$ are isomorphic.*

Proof. Let $\gamma \neq 0$ be such that $H^0(X, \mathcal{O}_X(E))_\gamma^{(B)} \neq \emptyset$. Since $E \in (\partial_G X)^{n\ell}$, the character γ is non-negative on $\rho_{G,X}((\partial_G X)^\ell)$, which generates the whole $V_G^\ell(X)$ as a convex cone, because X is complete. It follows that $\gamma \in V_G^\ell(X)$, which implies that some positive integral multiple of γ , say $n\gamma$, lies in $\Lambda_G(\mathbb{X})$. Let us also assume that it is indecomposable in $\Lambda_G(\mathbb{X})$, i.e. that n is minimal satisfying $n > 0$ and $n\gamma \in \Lambda_G(X)$.

Consider $\pi(E)$: if it is not a G -stable prime divisor of \mathbb{X} , then $\pi_*(\rho_{G,X}(E))$ is in $V_G(\mathbb{X})$ but doesn't lie on any 1-dimensional face of $V_G(\mathbb{X})$. On the other hand, each element of $\partial_G \mathbb{X}$ is the image $\pi(D)$ of some G -stable prime divisor D of X , with $\pi_*(\rho_{G,X}(D))$ equal to a positive rational multiple of $\rho_{G,\mathbb{X}}(\pi(D))$. This implies that $n\gamma \in \Lambda_G(\mathbb{X})$ is non-negative on $\rho_{G,\mathbb{X}}(\partial_G \mathbb{X})$ and negative on $\pi_*(\rho_{G,X}(E))$, which is absurd because $\rho_{G,\mathbb{X}}(\partial_G \mathbb{X})$ generates $V_G(\mathbb{X})$ as a convex cone.

We conclude that $\pi(E) \in \partial_G \mathbb{X}$, and that E is the unique element of $\partial_G X$ whose image is $\pi(E)$, because $n\gamma$ is non-negative on $\rho_{G,X}(E')$ for any $E' \in \partial_G X$ different from E . Let $0 > -m = \langle \rho_{G,\mathbb{X}}(\pi(E)), n\gamma \rangle$. Then $H^0(\mathbb{X}, \mathcal{O}_{\mathbb{X}}(m\pi(E))) \neq \mathbb{C}$.

From [Br07, Theorem 2.2.3] it follows that $H^0(\mathbb{X}, \mathcal{O}_{\mathbb{X}}(\pi(E))) \neq \mathbb{C}$, that $H^0(\mathbb{X}, \mathcal{O}_{\mathbb{X}}(\pi(E)))/\mathbb{C}$ is irreducible with highest weight $\sigma_{\pi(E)}$, and that any $\chi \in \Lambda_G(\mathbb{X})$ satisfying

$$(7.2) \quad \langle \rho_{G,\mathbb{X}}(D), \chi \rangle \geq 0 \quad \forall D \in (\partial_G \mathbb{X} \setminus \{\pi(E)\}) \cup \Delta_G(\mathbb{X}), \quad \langle \rho_{G,\mathbb{X}}(\pi(E)), \chi \rangle < 0$$

is a positive multiple of $\sigma_{\pi(E)}$. We have then shown (1). It also follows that $n\gamma$ is a positive multiple of $\sigma_{\pi(E)}$, whence γ is non-positive on $V_G(X)$ and so it is zero on $\rho_{G,X}(D)$ for all $D \in \partial_G X$ different from E . This shows (7.1).

Now recall that $n\gamma$ is indecomposable in $\Lambda_G(\mathbb{X})$. Since it is a positive multiple of $\sigma_{\pi(E)}$, it is equal to $\sigma_{\pi(E)}$. On the other hand $\Sigma_G(X) = \Sigma_G(\mathbb{X})$ and $\sigma_{\pi(E)}$ is also indecomposable in $\Lambda_G(X)$. Therefore $n = 1$, and we have

$$\langle \rho_{G,X}(E), \gamma \rangle = \langle \pi_*(\rho_{G,X}(E)), \gamma \rangle = -1 = \langle \rho_{G,\mathbb{X}}(\pi(E)), \sigma_{\pi(E)} \rangle,$$

whence $\pi_*(\rho_{G,X}(E)) = \rho_{G,\mathbb{X}}(\pi(E))$. The proof of part (2) is complete.

Since γ is the highest weight of an arbitrary non-trivial G -submodule of $H^0(X, \mathcal{O}_X(E))$, and the latter is multiplicity-free since X is spherical, the proof of (3) is also complete. \square

Definition 7.3. We denote by

$$\Lambda_G(X, \mathcal{E}) \subseteq \Lambda_G(X)$$

the sublattice generated by the elements $\sigma_{\pi(E)}$ for all $E \in \mathcal{E}$.

Corollary 7.4.

$$\Lambda_G(X) = \rho_{G,X}(\mathcal{E})^\perp \oplus \Lambda_G(X, \mathcal{E}).$$

Proof. From Lemma 7.2 we deduce that for all $E \in \mathcal{E}$ the element $\rho_{G,X}(E)$ is -1 on the spherical root $\sigma_{\pi(E)}$ of X , and zero on all other spherical roots of X . The corollary follows. \square

Remark 7.5. In the proof of Lemma 7.2 we used the crucial fact that X and $\mathbb{X}(G/\widehat{H})$ have the same spherical roots. The decomposition of $\Lambda(G/H)$ into the above direct sum would indeed be false in general, if we had used $\mathbb{X}(G/\overline{H})$ instead of $\mathbb{X}(G/\widehat{H})$.

Definition 7.6. Define

$$\mathbb{E} = \{\pi(E) \mid E \in \mathcal{E}\},$$

and

$$\mathbb{D} = \partial\mathbb{X} \setminus \mathbb{E}.$$

Definition 7.7. Let $A' = A'(X, \mathcal{D})$ be the universal cover of $\text{Aut}^\circ(\mathbb{X}, \mathbb{D})$, and $A = A(X, \mathcal{D}) = A' \times C$. We denote by

$$\vartheta': G' \rightarrow A'$$

the lift of $\theta_{G',\mathbb{X}}: G' \rightarrow \text{Aut}^\circ(\mathbb{X}, \mathbb{D})$ to A' , and we set

$$\vartheta = \vartheta' \times \text{id}_C: G \rightarrow A.$$

We also choose a Borel subgroup B_A of A such that $B_A \supseteq \vartheta(B)$.

Now we are ready to state the main result of this section.

Theorem 7.8. *The action of $A(X, \mathcal{D})$ lifts from \mathbb{X} to X , and the image of $A(X, \mathcal{D})$ inside $\text{Aut}^\circ(X)$ is equal to $\text{Aut}^\circ(X, \mathcal{D})$. As an $A = A(X, \mathcal{D})$ -variety, X is G -regular with boundary \mathcal{D} . The vector space $N_A(X)$ is naturally identified with $\Lambda_G(X, \mathcal{E})^\perp \subseteq N_G(X)$. The fan $\mathcal{F}_A(X)$ of X as an A -variety is given by intersecting all cones of $\mathcal{F}_G(X)$ with $\Lambda_G(X, \mathcal{E})^\perp$.*

The proof occupies the rest of the section: the theorem follows from Lemma 7.13, Theorem 7.18, and Corollary 7.20.

Example 7.9. It is necessary to define A' as the universal cover of $\text{Aut}^\circ(\mathbb{X}, \mathbb{D})$. Consider for example $G = \text{SL}(n+1)$ (with $n \geq 1$) acting linearly and diagonally on $\mathbb{P}^{n+1} \times (\mathbb{P}^n)^*$, where on the first factor it acts only on the first $n+1$ homogeneous coordinates. Then $X = \text{Bl}_p(\mathbb{P}^{n+1}) \times (\mathbb{P}^n)^*$, with $p = [0, \dots, 0, 1]$, is a G -regular variety with three G -stable prime divisors, of which only one lies in $(\partial_G X)^{n\ell}$. We have $\mathbb{X} = \mathbb{P}^n \times (\mathbb{P}^n)^*$, and if $\mathcal{D} = (\partial X)^\ell$ then $\mathbb{D} = \emptyset$. The action of $\text{Aut}^\circ(\mathbb{X}, \mathbb{D}) = \text{Aut}^\circ(\mathbb{X}) = \text{PGL}(n+1) \times \text{PGL}(n+1)$ doesn't lift to X , whereas the action of its universal cover does.

In view of proving Theorem 7.8, we start finding a candidate for a generic stabilizer of the A -action on X . Let $\widehat{H}_A \subseteq A$ be² the stabilizer of the point $e\widehat{H} \in G/\widehat{H} \subseteq \mathbb{X}$. The colors of \mathbb{X} as a G -variety and as an A -variety coincide, thanks to [Br07, Theorem 2.4.2], and we have $\vartheta(\widehat{H}) = \widehat{H}_A \cap \vartheta(G)$.

We also notice that thanks to our general assumptions any G -linearization of an invertible sheaf \mathbb{X} can be uniquely extended to an A -linearization, inducing an identification of the two groups $\text{Pic}^G(\mathbb{X})$ and $\text{Pic}^A(\mathbb{X})$.

Lemma 7.10. (1) *The pull-back of characters of B_A along $\vartheta|_B$ induces an injective map $r: \Lambda_A(\mathbb{X}) \rightarrow \Lambda_G(\mathbb{X})$. It maps $\Sigma_A(\mathbb{X})$ to the set of spherical roots $\{\sigma_D \mid D \in \mathbb{D}\}$.*
 (2) *The dual map $r^*: N_G(\mathbb{X}) \rightarrow N_A(\mathbb{X})$ satisfies*

$$r^*(V_G(\mathbb{X})) = V_A(\mathbb{X}).$$

(3) *We have that $\partial_A \mathbb{X} = \mathcal{D}$, and*

$$A/\widehat{H}_A = \mathbb{X} \setminus \bigcup_{D \in \mathbb{D}} D.$$

(4) *The pull-back of characters of \widehat{H}_A along $\vartheta|_{\widehat{H}}$ is a surjective homomorphism $r': \mathcal{X}(\widehat{H}_A) \rightarrow \mathcal{X}(\widehat{H})$ with free kernel of rank $|\mathbb{E}|$.*

Proof. The injectivity of r is obvious, since it corresponds to taking a B_A -semiinvariant $f \in \mathbb{C}(\mathbb{X})$ and considering it as a B -semiinvariant. The rest of part (1) follows from the results of [Pe09], and it can also be shown directly using the following fact: the spherical roots of \mathbb{X} are the T -weights appearing in the quotient of tangent spaces

$$\frac{T_z \mathbb{X}}{T_z(Gz)},$$

where $z \in \mathbb{X}$ is the unique fixed point of B^- . Let us choose a maximal torus T_A of A containing $\vartheta(T)$: if $B_A^- \subseteq A$ is the Borel subgroup satisfying $B_A \cap B_A^- = T_A$ then B_A^- contains $\vartheta(B^-)$.

²Our notation is consistent thanks to Corollary 7.14.

Hence z is also the unique B_A^- -fixed point, therefore the spherical roots of \mathbb{X} as an A -variety are the T_A -weights appearing in the quotient of tangent spaces

$$\frac{T_z \mathbb{X}}{T_z(Az)},$$

form the set $\Sigma_A(\mathbb{X})$. This implies part (1), and part (2) is an immediate consequence.

The first statement of part (3) stems from the fact that each $E \in \mathbb{E}$ is not stable under the action of A , and the second follows from the first because \mathbb{X} is wonderful under the action of A .

For part (4), we notice that r' can be identified with the natural map

$$\frac{\mathcal{X}(C) \times \mathbb{Z}^\Delta}{\bar{\rho}_{A,\mathbb{X}}(\Lambda_A(\mathbb{X}))} \rightarrow \frac{\mathcal{X}(C) \times \mathbb{Z}^\Delta}{\bar{\rho}_{G,\mathbb{X}}(\Lambda_G(\mathbb{X}))}$$

(see diagram (3.1)). The kernel of r' is then $\Lambda_G(\mathbb{X})/r(\Lambda_A(\mathbb{X}))$ which is free, generated by the spherical roots σ_E for all $E \in \mathbb{E}$ by part (1). \square

Let us put together two copies of the diagram (3.1), one for the G - and one for the A -action, also adding the extensions of $\bar{\rho}_G$ and $\bar{\rho}_A$ resp. to $\Lambda_G(G/H)$ and $\Lambda_A(A/\widehat{H}_A)$, as in §3. We obtain a commutative diagram

$$\begin{array}{ccccccc} \Lambda_A(A/K_A) \hookrightarrow \Lambda_A(A/\widehat{H}_A) & \xrightarrow{\bar{\rho}_A} & \mathcal{X}(C) \times \mathbb{Z}^\Delta & \xrightarrow{\tau_A} & \mathcal{X}(K_A) & \twoheadrightarrow & \mathcal{X}(\widehat{H}_A) \\ \downarrow r & & \parallel & \nearrow \sigma_A & \downarrow (\vartheta|_B)^* & & \downarrow r' \\ \Lambda_G(G/K) \hookrightarrow \Lambda_G(G/\widehat{H}) & \hookrightarrow \Lambda_G(G/H) & \xrightarrow{\bar{\rho}_G} & \mathcal{X}(C) \times \mathbb{Z}^\Delta & \xrightarrow{\tau_G} & \mathcal{X}(K) & \twoheadrightarrow \mathcal{X}(\widehat{H}) \\ & & & \nearrow \sigma_G & & & \end{array}$$

where K_A is the spherical closure of \widehat{H}_A , and K is the spherical closure of \widehat{H} (and of H). The last arrow of the first row is the restriction map, which can be seen as the quotient

$$\mathcal{X}(K_A) \rightarrow \mathcal{X}(K_A)/\mathcal{X}(K_A)^{\widehat{H}_A} \cong \mathcal{X}(\widehat{H}_A).$$

The same remark holds for the last map of the second row and the groups K, \widehat{H} .

In order to determine a generic stabilizer in A for X , we start defining a lattice $\Lambda \subseteq \Lambda_G(G/H)$. A posteriori, it will be the lattice of B -eigenvalues χ_f of B_A -eigenvectors $f \in \mathbb{C}(X)^{(B_A)}$. Such a function f cannot have zeros nor poles on the divisors in \mathcal{E} , since these are not A -stable, nor are A -colors of X . This suggests the definition of Λ given in the following.

Definition 7.11. Let Λ be the lattice

$$\Lambda = \rho_{G,X}(\mathcal{E})^\perp \subseteq \Lambda_G(G/H).$$

Proposition 7.12. *The following inclusion holds:*

$$\bar{\rho}_G(\Lambda) \supseteq \bar{\rho}_A(\Lambda_A(A/K_A)).$$

The subgroup H_A of K_A corresponding to the lattice $\bar{\rho}_G(\Lambda)$ is a spherical subgroup of A , and we have $\vartheta(H) = H_A \cap \vartheta(G)$. This induces a G -equivariant identification of G/H with an open subset of A/H_A .

Proof. Let $\chi \in \Lambda_A(A/K_A) \subseteq \Lambda_A(A/\widehat{H}_A)$. If $f \in \mathbb{C}(A/\widehat{H}_A)_\chi^{(B_A)}$, then consider its pull-back on X , denoted by \tilde{f} . It is also a B -eigenvector with B -eigenvalue $\tilde{\chi} = r(\chi)$.

We know that the divisor $\text{div}(\tilde{f})$ on X is B_A -stable, so in general it is a linear combination of colors and A -stable prime divisors. In any case, its components do not belong to \mathcal{E} , because the latter consists of prime divisors moved by A . It follows that all discrete valuations in $V_G(G/H)$ coming from these elements of \mathcal{E} must take the value 0 on $\tilde{\chi}$.

Therefore $\tilde{\chi} \in \Lambda$, and the first assertion is proved. In order to verify that H_A is spherical we have to show that σ_A restricted to $\bar{\rho}_G(\Lambda) = \tau_A^{-1}(\mathcal{X}(K_A)^{H_A})$ is injective. But we already know that the restriction of σ_G on $\bar{\rho}_G(\Lambda_G(G/H))$ is injective, and that $\Lambda \subseteq \Lambda_G(G/H)$: this proves the second assertion.

Next, we claim that r' induces an isomorphism between $\mathcal{X}(\widehat{H}_A)^{H_A}$ and $\mathcal{X}(\widehat{H})^H$. This shows that $\widehat{H}_A/H_A \cong \widehat{H}/H$, and the rest of the lemma follows. To prove the claim, it is enough to notice that

$$\begin{aligned} \mathcal{X}(\widehat{H}_A)^{H_A} &\cong \frac{\bar{\rho}_G(\Lambda)}{\ker \tau_A} \\ &= \frac{\bar{\rho}_G(\Lambda)}{\bar{\rho}_G(r(\Lambda_A(A/\widehat{H}_A)))} \\ &\cong \frac{\bar{\rho}_G(\Lambda) \oplus \bar{\rho}_G(\Lambda_G(X, \mathcal{E}))}{\bar{\rho}_G(\Lambda_G(G/\widehat{H}))} \\ &= \frac{\bar{\rho}_G(\Lambda \oplus \Lambda_G(X, \mathcal{E}))}{\bar{\rho}_G(\Lambda_G(G/\widehat{H}))} \\ &= \frac{\bar{\rho}_G(\Lambda_G(G/H))}{\bar{\rho}_G(\Lambda_G(G/\widehat{H}))} \\ &\cong \mathcal{X}(\widehat{H})^H, \end{aligned}$$

and that the resulting isomorphism $\mathcal{X}(\widehat{H}_A)^{H_A} \cong \mathcal{X}(\widehat{H})^H$ is indeed induced by r' . \square

We build *ex novo* an embedding X_A of A/H_A , and then prove that we actually obtain X .

Lemma 7.13. *The pull-back of characters of B_A to B along $\vartheta|_B$ induces an injective map $s: \Lambda_A(A/H_A) \rightarrow \Lambda_G(G/H)$ whose image is Λ . The dual map $s^*: N_G(G/H) \rightarrow N_A(A/H_A)$ satisfies*

$$s^*(V_G(G/H)) = V_A(A/H_A),$$

and induces an isomorphism

$$s^*|_{V_G^\ell(G/H)}: V_G^\ell(G/H) \rightarrow V_A^\ell(A/H_A).$$

Proof. Let $\gamma \in \Lambda_A(A/H_A)$: it is the B_A -eigenvalue of a B_A -eigenvector $f \in \mathbb{C}(A/H_A)^{(B_A)}$. But f is a B -eigenvector too and the character $\chi = s(\gamma)$ is its B -eigenvalue. Both the B - and the B_A -eigenvalue determine f up to a multiplicative constant, hence s is injective.

Consider the commutative diagram

$$\begin{array}{ccc} \Lambda_A(A/H_A) & \xhookrightarrow{\bar{p}_A} & \text{Pic}^A(\mathbb{X}) \\ \downarrow s & & \parallel \\ \Lambda_G(G/H) & \xhookrightarrow{\bar{p}_G} & \text{Pic}^G(\mathbb{X}) \end{array}$$

From the definition of H_A we have $\bar{p}_A(\Lambda_A(A/H_A)) = \bar{p}_G(\Lambda)$, therefore we obtain $s(\Lambda_A(A/H_A)) = \Lambda$.

Let $v \in V_A(A/H_A)$. It corresponds to an A -invariant valuation, which is *a fortiori* G -invariant too: in other words we can compute v also on $\Lambda_G(G/H)$ obtaining an element of $V_G(G/H)$. This shows that $s^*(V_G(G/H)) \supseteq V_A(A/H_A)$.

Then we notice that s extends the map r of Lemma 7.10. This gives the commutative diagram

$$\begin{array}{ccc} N_G(G/H) & \xrightarrow{\pi_{*, \widehat{H}}} & N_G(G/\widehat{H}) \\ \downarrow s^* & & \downarrow r^* \\ N_A(A/H_A) & \xrightarrow{\pi_{*, \widehat{H}_A}} & N_A(A/\widehat{H}_A) \end{array}$$

where $V_A(A/H_A)$ (resp. $V_G(G/H)$) is the inverse image of $V_A(A/\widehat{H}_A)$ (resp. $V_G(G/\widehat{H})$) thanks to Lemma 3.2.

This, together with Lemma 7.10, part (2), proves $s^*(V_G(G/H)) = V_A(A/H_A)$. The image of $V_G^\ell(G/H)$ is contained in $V_A^\ell(A/H_A)$, and we conclude the proof observing that the dimensions of $V_G^\ell(G/H)$ and $V_A^\ell(A/H_A)$ are both equal to the dimension of $\widehat{H}/H \cong \widehat{H}_A/H_A$. \square

Corollary 7.14. *The wonderful closure of H_A is \widehat{H}_A .*

Proof. By construction $H_A \subseteq \widehat{H}_A \subseteq K_A = \overline{H_A}$. From Lemma 7.13 we deduce that A/H_A and A/\widehat{H}_A have the same spherical roots: the corollary follows then from Proposition 3.10. \square

We shall now define the fan of convex cones of X_A , using that of X . First, we collect some consequences on $\mathcal{F}(X)$ of the analysis we have developed so far.

Definition 7.15. Let \mathcal{F} be a fan of convex cones, consider a subset $\mathcal{F}' \subset \mathcal{F}$ and let $c \in \mathcal{F} \setminus \mathcal{F}'$ be 1-dimensional. Then \mathcal{F} is the *join* of \mathcal{F}' and c if each element of $\mathcal{F} \setminus \mathcal{F}'$ is the convex cone generated by c and an element of \mathcal{F}' .

- Corollary 7.16.** (1) Let $E \in \mathcal{E}$, and let $\mathcal{F}_G^{\sigma_{\pi(E)}}(X)$ be the fan of convex cones obtained intersecting each element of $\mathcal{F}_G(X)$ with $\sigma_{\pi(E)}^\perp$. Then $\mathcal{F}_G(X)$ is the join of $\mathcal{F}_G^{\sigma_{\pi(E)}}(X)$ and $c_{X,E}$.
- (2) Let $\mathcal{F}_G^\Lambda(X)$ be the fan of convex cones obtained intersecting each element of $\mathcal{F}_G(X)$ with $\Lambda_G(X, \mathcal{E})^\perp$. Then the restriction of s^* to $\text{supp } \mathcal{F}_G^\Lambda(X)$ is injective, and $s^*(\text{supp } \mathcal{F}_G^\Lambda(X)) = V_A(A/H_A)$.
- (3) The set

$$\{s^*(c) \mid c \in \mathcal{F}_G^\Lambda(X)\}$$

is a fan of polyhedral convex cones in $N_A(A/H_A)$. The associated embedding of A/H_A is smooth and complete.

Proof. Part (1) follows from Lemma 7.2, part (2). Part (2) follows from part (1) applied to all $E \in \mathcal{E}$, together with Corollary 7.4 and Lemma 7.13. We turn to part (3). Completeness of this embedding is an immediate consequence of part (2). For smoothness, we observe that a maximal cone c of $\mathcal{F}_G(X)$ can be written as

$$c = (\{-\sigma_E \mid E \in \mathcal{E}\} \cup \Psi)^\vee$$

where Ψ is a basis of $\Lambda = \rho_{G,X}(\mathcal{E})^\perp$, thanks to the smoothness of X together with part (1) applied to all $E \in \mathcal{E}$ and Corollary 7.4. Therefore

$$s^*(c \cap (\Lambda_G(X, \mathcal{E})^\perp)) = (s^{-1}(\Psi))^\vee.$$

The smoothness characterization recalled in §2 is verified, since $s^{-1}(\Psi)$ is a basis of $\Lambda_A(A/H_A)$, and the proof is complete. \square

Definition 7.17. We define

$$\mathcal{F}_A = \{s^*(c) \mid c \in \mathcal{F}_G^\Lambda(X)\},$$

and we denote by X_A the corresponding embedding of A/H_A .

Theorem 7.18. The inclusion $G/H \subseteq A/H_A$ extends to an A -equivariant isomorphism between X and X_A .

Proof. The group G acts on X_A via the map θ , and it is enough to show X_A is a toroidal embedding of G/H with fan $\mathcal{F}_G(X)$. Let us first prove the theorem with the assumption that $|\mathcal{E}| = 1$, say $\mathcal{E} = \{E\}$.

In addition to the G -equivariant map $\pi: X \rightarrow \mathbb{X}$ we also have by construction an A -equivariant map $\pi_A: X_A \rightarrow \mathbb{X}$ extending the projection $\pi^{H_A, \hat{H}_A}: A/H_A \rightarrow A/\hat{H}_A$. The A -colors and the G -colors of \mathbb{X} coincide, and this implies the same for X_A : indeed any A -color (resp. G -color) of X_A is of the form $\pi_A^{-1}(D)$ for an A -color (resp. G -color) D of \mathbb{X} .

If $D \subset \mathbb{X}$ is a color such that $\pi_A^{-1}(D)$ contains a G -orbit $Y \subset X_A$, then D contains the G -orbit $\pi_A(Y)$: this is absurd because \mathbb{X} is a toroidal G -variety. In other words X_A is a toroidal G -variety.

Next, we claim that A/H_A is a G -embedding of G/H whose fan contains $c_{X,E}$ as its unique non-trivial cone. Part (3) of Lemma 7.10 implies that A/\widehat{H}_A is an elementary embedding of G/\widehat{H} , with orbits G/\widehat{H} , $\pi(E) \cap A/\widehat{H}_A$, and fan containing $c_{\mathbb{X},\pi(E)}$ as its unique non-trivial cone. The open subset $G/H \subset A/H_A (\subseteq X_A)$ is equal to $\pi_A^{-1}(G/\widehat{H})$, and the G -stable closed subset $E' = (A/H_A) \setminus (G/H)$ is equal to $\pi_A^{-1}(\pi(E)) \cap A/H_A$.

Consider the G -invariant prime divisors contained in E' : they are neither colors nor A -stable prime divisors. We claim that there is only one of them, with associated convex cone $c_{X,E}$. Then E' itself is a G -stable prime divisor, because we already proved that A/H_A is a toroidal embedding of G/H .

For this, consider $f \in \mathbb{C}(G/H)_\lambda^{(B)}$ with $\lambda \in \Lambda$. By Lemma 7.13 we have that f is also a B_A -eigenvector, therefore its divisor $\text{div}(f)$ on A/H_A has components which are either colors or A -stable prime divisors. It follows that $\rho_{G,A/H_A}(F) \in \lambda^\perp$ for all $\lambda \in \Lambda$ and all G -stable prime divisor $F \subseteq E'$. Since $c_{X,E} = \Lambda^\perp \cap V_G(G/H)$, we deduce that there is only one such F and it satisfies $\rho_{G,A/H_A}(F) \in c_{X,E}$: the claim above follows.

Now Lemma 7.2, Lemma 7.13 and Corollary 7.16 part (1) hold also if we replace X with X_A and \mathcal{D} with the set $(\partial_G X_A) \setminus \{E'\}$. From Corollary 7.16 part (1) we deduce that $\mathcal{F}_G(X_A)$ is the join of $\mathcal{F}_G^{\sigma_{\pi(E)}}(X_A)$ and $c_{X,E}$. From Lemma 7.13 we deduce that every G -stable prime divisor D of X_A such that $\rho_{G,X_A}(D) \in \sigma_{\pi(E)}^\perp$ is also A -stable, hence each G -orbit $Y \subseteq X_A$ such that $c_{X_A,Y} \subset \sigma_{\pi(E)}^\perp$ is also an A -orbit.

In other words $\mathcal{F}_G^{\sigma_{\pi(E)}}(X_A)$ and $\mathcal{F}_G^{\sigma_{\pi(E)}}(X)$ have the same image under s_* , which implies that they are equal. The theorem in the case $|E| = 1$ follows.

If $|\mathcal{E}| > 1$, we consider the chain of groups

$$\theta_{G,X}(G) \subseteq \text{Aut}^\circ(X, \partial_G X \setminus \{E_1\}) \subseteq \text{Aut}^\circ(X, \partial_G X \setminus \{E_1, E_2\}) \subseteq \dots \subseteq \text{Aut}^\circ(X, \mathcal{D}),$$

where $\mathcal{E} = \{E_1, E_2, \dots\}$, and proceed by induction on $|\mathcal{E}|$. Let $A_i \subseteq A_{i+1}$ be two consecutive groups of this chain: we may apply the first part of the proof, together with Corollary 7.19 below (whose proof in the case $|\mathcal{E}| = 1$ only depends on the case $|\mathcal{E}| = 1$ of this theorem) to the A_i -variety X . We obtain the construction of an A_{i+1} -variety $X_{A_{i+1}}$, which is A_i -equivariantly isomorphic to X . \square

Corollary 7.19. *We have $\partial_A X = \mathcal{D}$ and $(\partial_A X)^\ell = (\partial_G X)^\ell$.*

Proof. This is obvious from the definition of \mathcal{F}_A . \square

Corollary 7.20. *The image of A in $\text{Aut}^\circ(X)$ is equal to $\text{Aut}^\circ(X, \mathcal{D})$.*

Proof. By construction A moves each element of \mathbb{E} on \mathbb{X} and stabilizes all elements of \mathbb{D} , hence $\mathbb{D} = \partial_A \mathbb{X}$.

Moreover \widehat{H}_A is the wonderful closure of H_A , hence we can apply the exact sequence (4.2) to X as an A -variety, mapping onto the wonderful A -variety \mathbb{X} . Since the image of A contains by construction both the universal cover of $\text{Aut}^\circ(\mathbb{X}, \mathbb{D}) = \text{Aut}^\circ(\mathbb{X}, \partial_A \mathbb{X})$ and $(\widehat{H}_A/H_A)^\circ \cong (\widehat{H}/H)^\circ \subseteq C$, it follows that the image of A contains $\text{Aut}^\circ(X, \partial_A X) = \text{Aut}^\circ(X, \mathcal{D})$. \square

8. ABELIAN CASE

In this section we will assume that $G = C$ is an algebraic torus, X as usual a complete G -regular variety, and $\mathcal{D} \subseteq \partial_G X$ any subset. Hence X is a toric variety under the action of a quotient of G . Since G is equal to its own Borel subgroups, X has no G -color.

We recall now the description of $\text{Aut}^\circ(X)$ given in [Oda88]. In this setting the study of $\text{Aut}^\circ(X)$ is simplified by the fact that, for all $D \in \partial_G X$, the G -module $H^0(X, \mathcal{O}_X(D))$ splits into the sum of 1-dimensional G -submodules.

Definition 8.1. Suppose that for some non-zero $\alpha \in \Lambda_G(X)$ the divisor $X(\alpha)$ exists, i.e. that there exist $X(\alpha) \in \partial_G X$ and an element $f_\alpha \in H^0(X, \mathcal{O}_X(X(\alpha)))_\alpha^{(B)}$. Then we denote by $u_\alpha: \mathbb{C} \rightarrow \text{Aut}^\circ(X)$ the unipotent 1-PSG corresponding to α defined in [Oda88, Proposition 3.14], and such that $X(\alpha)$ is the unique G -stable prime divisor not stable under $U_\alpha = u_\alpha(\mathbb{C})$.

We recall that [Oda88, Proposition 3.14] gives explicit formulae for u_α , and that this 1-PSG can also be defined in the following way. The element $\alpha \in \Lambda_G(X)$ naturally corresponds to a semisimple 1-PSG of $\text{Aut}^\circ(X)$ through the action of G on X . Denote by δ_α its derivative, which is a tangent vector field on X . Then the tangent vector field du_α is equal to $f_\alpha \delta_\alpha$.

Remark 8.2. If $X(\alpha)$ exists for some α , then $\langle \rho_{G,X}(X(\alpha)), \alpha \rangle = -1$ and $\langle \rho_{G,X}(D), \alpha \rangle \geq 0$ for all $D \in \partial_G X$ different from $X(\alpha)$. However, the difference in signs from our discussion and [Oda88, §3.4] is only apparent: a character $\lambda \in \mathcal{X}(\theta_{G,X}(G))$ is indeed a rational function on X and a G -eigenvector, but of G -eigenvalue $-\lambda$.

Notice that the assignment $\alpha \mapsto X(\alpha)$ might be not injective. Also, if both $X(\alpha)$ and $X(-\alpha)$ exist, then $\rho_{G,X}(X(\alpha))$ is not necessarily $-\rho_{G,X}(X(-\alpha))$. However, $X(\alpha)$ and $X(-\alpha)$ are the only G -stable prime divisors whose images through $\rho_{G,X}$ are non-zero on α .

Definition 8.3. Let $\mathcal{D} \subseteq \partial_G X$ any subset, and define $\Phi = \Phi(X, \mathcal{D})$ to be the maximal set of roots of X such that:

- (1) if $\alpha \in \Phi(X, \mathcal{D})$ then also $-\alpha \in \Phi(X, \mathcal{D})$;
- (2) if $\alpha \in \Phi(X, \mathcal{D})$ then $X(\alpha) \in \mathcal{E} = \partial X \setminus \mathcal{D}$.

The following result is an immediate consequence of [Oda88, Demazure's Structure Theorem, §3.4].

Theorem 8.4. *The subgroup of $\text{Aut}^\circ(X)$ generated by $\theta_{G,X}(G)$ and U_α for all $\alpha \in \Phi(X, \mathcal{D})$ has $\Phi(X, \mathcal{D})$ as root system with respect to its maximal torus $\theta_{G,X}(G)$, and is a Levi subgroup of $\text{Aut}^\circ(X, \mathcal{D})$.*

Definition 8.5. Define $A = A(X, \mathcal{D})$ the subgroup of $\text{Aut}^\circ(X)$ generated by $\theta_{G,X}(G)$ and U_α for all $\alpha \in \Phi(X, \mathcal{D})$. Let us also choose a Borel subgroup $B_A \subseteq A$ containing G and, consequently, a subdivision of Φ into positive and negative roots, resp. denoted by $\Phi_+ = \Phi_+(X, \mathcal{D})$ and $\Phi_- = \Phi_-(X, \mathcal{D})$, and denote by $\Psi = \Psi(X, \mathcal{D})$ the basis of positive roots.

Since B_A is generated by $\theta_{G,X}(G)$ together with the subgroups U_α for all $\alpha \in \Psi$, we have that any G -stable prime divisor which doesn't appear as $X(\alpha)$ for some $\alpha \in \Psi$ is B_A -stable. In other words

$$(8.1) \quad \{X(\alpha) \mid \alpha \in \Phi_+\} = \{X(\alpha) \mid \alpha \in \Psi\},$$

and for the same reason (replacing Ψ with $-\Psi$)

$$(8.2) \quad \{X(\alpha) \mid \alpha \in \Phi_-\} = \{X(\alpha) \mid \alpha \in (-\Psi)\}.$$

Lemma 8.6. *Let $\alpha, \beta \in \Phi$, and suppose that $X(\alpha) = X(\beta)$. Then $\gamma = \alpha - \beta$ and $-\gamma$ are also in Φ , with $X(\gamma) = X(-\beta)$ and $X(-\gamma) = X(-\alpha)$.*

Proof. Suppose that $X(-\alpha) = X(-\beta)$. Then $\alpha - \beta$ is zero on $\rho_{G,X}(X(\pm\alpha))$ and on $\rho_{G,X}(X(\pm\beta))$. On the other hand, if a G -stable prime divisor $D \subset X$ is not of the form $X(\pm\alpha)$ nor $X(\pm\beta)$, then both α and β are zero on $\rho_{G,X}(D)$. It follows that $\text{supp } \mathcal{F}_G(X)$ is contained in the hyperplane $(\alpha - \beta)^\perp$ of $N_G(X)$, which contradicts the completeness of X . Therefore $X(-\alpha) \neq X(-\beta)$, i.e. $X(\alpha)$, $X(-\alpha)$ and $X(-\beta)$ are three different prime divisors. The statement of the lemma is now obvious. \square

Lemma 8.7. *The matrix*

$$(8.3) \quad (\langle \rho_{G,X}(X(\alpha)), \alpha \rangle)_{\alpha \in \Psi}$$

is non-degenerate. In particular, the elements $\rho_{G,X}(X(\alpha))$, for α varying in Ψ , are linearly independent.

Proof. Thanks to Lemma 8.6, the elements $\rho_{G,X}(X(\alpha))$ for $\alpha \in \Psi$ are pairwise distinct. If the matrix (8.3) is degenerate, there exists a linear combination

$$(8.4) \quad \sum_{\alpha \in \Psi'} a_\alpha \rho_{G,X}(X(\alpha)) \in \Psi^\perp$$

where $\emptyset \neq \Psi' \subseteq \Psi$ and $a_\alpha \neq 0$ for all $\alpha \in \Psi'$. Applying $\langle -, \alpha \rangle$ for a fixed $\alpha \in \Psi'$ to the linear combination (8.4), we see that both $\rho_{G,X}(X(\alpha))$ and $\rho_{G,X}(X(-\alpha))$ must appear in the sum. Indeed, the former appears, and the latter is the only other possible summand that is nonzero on α . The elements $\rho_{G,X}(X(-\alpha))$ for $\alpha \in \Psi$ are distinct, thanks to the first part of the proof applied to the set of simple roots $-\Psi$.

Hence each summand in (8.4) can also be rewritten as $a_\alpha \rho_{G,X}(X(-\tau(\alpha)))$ where $\tau: \Psi' \rightarrow \Psi'$ is a bijection. We also know that $\rho_{G,X}(X(\alpha)) \neq \rho_{G,X}(X(-\alpha))$, therefore τ has no fixed points. Now consider

$$\gamma = \sum_{\alpha \in \Psi'} \alpha.$$

Its value on $\rho_{G,X}(D)$ is zero, if $D \subset X$ is a G -stable prime divisor not of the form $X(\pm\alpha)$ for some $\alpha \in \Psi'$. On the other hand, for a fixed $\alpha \in \Psi'$ we have that $X(\alpha) = X(-\tau(\alpha))$, but $X(\alpha) \neq X(\beta)$ for all $\beta \in \Psi$ different from α , and $X(\alpha) \neq X(-\beta)$ for any $\beta \in \Psi$ different from $\tau(\alpha)$. Therefore

$$\begin{aligned} \langle \rho_{G,X}(X(\alpha)), \gamma \rangle &= \langle \rho_{G,X}(X(\alpha)), \alpha \rangle + \langle \rho_{G,X}(X(\alpha)), \tau(\alpha) \rangle + \left\langle \rho_{G,X}(X(\alpha)), \sum_{\beta \in \Psi', \beta \neq \alpha, \tau(\alpha)} \beta \right\rangle \\ &= -1 + 1 + 0 = 0. \end{aligned}$$

We obtain that $\text{supp } \mathcal{F}_G(X)$ is contained in the hyperplane γ^\perp , which is absurd because X is complete. \square

Proposition 8.8. *As an A -variety, X is spherical (not necessarily toroidal). The set of its A -stable prime divisors is*

$$\partial_A X = \partial_G X \setminus \{X(\alpha) \mid \alpha \in \Phi\},$$

and these are exactly the G -stable prime divisors D such that $\rho_{G,X}(D) \in \Psi^\perp$. Given the identification $\mathcal{X}(\theta_{G,X}(G)) = \mathcal{X}(B_A)$, we have an inclusion

$$\iota: \Lambda_A(X) \rightarrow \Lambda_G(X)$$

whose image is the sublattice

$$(8.5) \quad \{\rho_{G,X}(X(\alpha)) \mid \alpha \in \Psi\}^\perp \subseteq \Lambda(X).$$

The restriction map $\iota^: \mathcal{N}_G(X) \rightarrow \mathcal{N}_A(X)$ induces an isomorphism*

$$\iota^*|_{\Psi^\perp}: \Psi^\perp \xrightarrow{\cong} \mathcal{N}_A(X).$$

For any B_A -stable prime divisor $D \subset X$ we have $\rho_{A,X}(D) = \iota^ \rho_{G,X}(D)$, and the set of A -colors of X is the following:*

$$\Delta_A(X) = \{X(\alpha) \mid \alpha \in (-\Psi)\} \setminus \{X(\alpha) \mid \alpha \in \Psi\}.$$

Finally, let $\alpha \in \Psi$ with $X(-\alpha) \in \Delta_A(X)$. For all $\beta \in \Phi_+$ different from α , we have $X(-\alpha) \neq X(\beta)$ and $X(\alpha) \neq X(\beta)$. In particular, if in addition $\beta \in \Psi$, we also have $\rho_{G,X}(X(-\alpha)) \in \beta^\perp$.

Proof. Since $\theta_{G,X}(G) \subseteq B_A$ has already an open orbit on X , the first statement is obvious. The statement about the A -stable prime divisors is also immediate.

Let us prove that the A -colors are the set $\Delta_A(X)$ as above defined. A color must be $X(\alpha)$ for some $\alpha \in \Phi$ otherwise it is A -stable, and at this point not being of the form $X(\alpha)$ for any $\alpha \in \Phi_+$ is equivalent to be stable under B_A . Then, we conclude using (8.1) and (8.2).

The inclusion ι is given by the simple observation that a B_A -eigenvector in $\mathbb{C}(X)$ is a fortiori a G -eigenvector, with same eigenvalue; the identity $\rho_{A,X}(D) = \iota^* \rho_{G,X}(D)$ for any B_A -stable prime divisor is also obvious.

Let us prove that the image of ι is the lattice (8.5). If $\gamma \in \Lambda_A(X)$, then a corresponding B_A -eigenvector $f_\gamma \in \mathbb{C}(X)$ cannot have zeros nor poles on prime divisors $X(\alpha)$ for $\alpha \in \Psi$, since the latter divisors are not B_A -stable. Hence $\iota(\Lambda_A(X)) \subseteq \{\rho_{G,X}(X(\alpha)) \mid \alpha \in \Psi\}^\perp$. On the other hand, if $\chi \in \{\rho_{G,X}(X(\alpha)) \mid \alpha \in \Psi\}^\perp$, then a corresponding G -eigenvector $f_\chi \in \mathbb{C}(X)$ has zeros and poles only on A -stable prime divisors or on colors. It follows that f_χ is also a B_A -eigenvector, and the other inclusion is proved.

We prove now that $\iota^*|_{\Psi^\perp}$ is an isomorphism between Ψ^\perp and $N_A(X)$. From the first part of the proof, this follows if we prove that

$$\Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = (\Psi \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus \left(\{\rho_{G,X}(X(\alpha)) \mid \alpha \in \Psi\}^\perp \otimes_{\mathbb{Z}} \mathbb{Q} \right),$$

and this equality is an easy consequence of Lemma 8.7.

Let us check the last statement, so let $\alpha \in \Psi$ be such that $X(-\alpha) \in \Delta_A(X)$, and consider $\beta \in \Phi_+$, $\beta \neq \alpha$. We know that $X(-\alpha) \neq X(\beta)$ because of the definition of $\Delta_A(X)$ together with (8.1). This also implies that $X(\alpha) \neq X(\beta)$, because otherwise we would have $\beta - \alpha \in \Phi_+$ with $X(-\alpha) = X(\beta - \alpha)$, thanks to Lemma 8.6. \square

Remark 8.9. The two above results imply in particular that the A -colors of X , seen as elements of $N_A(X)$, are linearly independent.

Example 8.10. An example where X is not toroidal as an A -variety can be given as follows. Let $X = \mathbb{P}^n$ with $n \geq 2$, under the linear action of the group G of $(n+1) \times (n+1)$ invertible diagonal matrices. Then $\partial_G X$ has $n+1$ elements, each given by the vanishing of an homogeneous coordinate. If \mathcal{D} is the set of all of them except for one, then $\text{Aut}^\circ(X, \mathcal{D})$ is a maximal parabolic subgroup of $\text{PGL}(n+1)$. Its Levi subgroup A containing the image of G acts with a fixed point, contained in all elements of \mathcal{D} therefore contained in any L -color of X .

We can now state the main theorem of this section.

Theorem 8.11. *If we identify $N_A(X)$ and Ψ^\perp via the map $\iota^*|_{\Psi^\perp}$ of Proposition 8.8, the fan of colored convex cones $\mathcal{F}_A(X)$ of X as a spherical A -variety is obtained from the fan $\mathcal{F}_G(X)$ as follows:*

$$\mathcal{F}_A(X) = \{(c \cap \Psi^\perp, d(c)) \mid c \in \mathcal{F}_G(X)\}.$$

Here $d(c)$ is the set of A -colors D of X such that if $\beta \in \Phi_+$ satisfies $X(-\beta) = D$, then both $\rho_{G,X}(X(\beta))$ and $\rho_{G,X}(X(-\beta))$ lie on 1-dimensional faces of c .

Proof. First, we consider $c \in \mathcal{F}_G(X)$ and we show that the colored cone $(c \cap \Psi^\perp, d(c))$ belongs to $\mathcal{F}_A(X)$.

The cone c is equal to $c_{X,Y}$ for some G -orbit Y . We claim that the colored cone associated to the A -orbit AY is given by $(c \cap \Psi^\perp, d(c))$, with $d(c)$ defined as in the theorem. To show the claim, it is enough to prove that:

- (1) the A -stable prime divisors containing AY are the G -stable prime divisors D such that $D \supseteq Y$ and $\rho_{G,X}(D) \in \Psi^\perp$;
- (2) the set of the A -colors containing AY is $d(c)$;
- (3) the convex cone c' generated by the image of elements of (1) and (2) under the map $\iota^* \circ \rho$ is $c \cap \Psi^\perp$.

Part (1) is obvious, thanks to the results on $\partial_A X$ contained in Proposition 8.8. For part (2), let us first prove that a color D not belonging to $d(c)$ doesn't contain AY . If D doesn't contain Y there is nothing to prove, therefore we may assume that $\rho_{G,X}(D)$ lies on a 1-dimensional face of c . Suppose at first that $X(-\beta) = D$ for some $\beta \in \Phi_+$, in such a way that $X(\beta)$ doesn't contain Y .

Let X_c be the affine G -stable open subset of X associated to the cone c , i.e.:

$$X_c = \{x \in X \mid \overline{Gx} \supseteq Y\}.$$

It is isomorphic to an affine space, and in [Oda88, Proof of Proposition 3.14] it is shown that X_c is stable under the action of $U_{-\beta}$.

More precisely, there exist global coordinates (x_1, \dots, x_n) on X_c such that $X(-\beta) \cap X_c$ is the hyperplane defined by the equation $x_1 = 0$, and in these coordinates $U_{-\beta}$ acts as follows:

$$(8.6) \quad u_{-\beta}(\xi)(x_1, x_2, \dots, x_n) = (x_1 + \xi, x_2, \dots, x_n).$$

One may easily check this formula using [Oda88, Proposition 3.14] and the fact that $X(-\beta)$ is the only G -stable prime divisor that contains Y and where β is non-zero. The hyperplane defined in X_c by $x_1 = 0$ contains Y , but from (8.6) we deduce that it doesn't contain $U_\beta Y$. As a consequence, AY is not contained in $X(-\beta)$.

Now we show that a color D in $d(c)$ contains AY . At first, consider $\beta \in \Phi_+$ such that $X(-\beta) = D$. Both $X(-\beta)$ and $X(\beta)$ contain Y , and we consider again the affine space X_c .

Applying [Oda88, Proposition 3.14] once again, there exist coordinates (x_1, x_2, \dots, x_n) such that $X(-\beta) \cap X_c$ is defined by the equation $x_1 = 0$, and $X(\beta) \cap X_c$ by the equation $x_2 = 0$, and such that

$$(8.7) \quad u_{-\beta}(\xi)(x_1, x_2, \dots, x_n) = (x_1 + \xi x_2, x_2, \dots, x_n)$$

and

$$(8.8) \quad u_{\beta}(\xi)(x_1, x_2, \dots, x_n) = (x_1, x_2 + \xi x_1, \dots, x_n).$$

We obtain that Y is both $U_{-\beta}$ -stable and U_{β} -stable, being contained in the subset of X_c defined by $x_1 = x_2 = 0$. Therefore $X(-\beta) = D$ contains $Y = U_{\beta}U_{-\beta}Y$.

Now observe that the image of the multiplication map

$$\theta_{G,X}(G) \times \prod_{\gamma \in \Phi} U_{\gamma} \rightarrow A$$

(where the product is taken in any fixed order) is dense in A . It follows that D contains AY , if we prove that D is U_{γ} -stable for all $\gamma \in \Phi$ such that $\gamma \neq \pm\beta$ for all $\beta \in \Phi_+$ satisfying $X(-\beta) = D$. For $\gamma \in \Phi_-$ there is nothing to prove. But also for $\gamma \in \Phi_+$ we know that $D \neq X(\gamma)$: this fact stems from the last statement of Proposition 8.8 together with (8.1). The proof of (2) is complete.

Let us prove (3). Call S the set of A -stable prime divisors containing AY . Then we can describe a minimal set of generators of c (as a convex cone) as the union of the following subsets:

- (a) the set $\rho_{G,X}(S)$;
- (b) for each color $D \in d(c)$, the set $\{\rho_{G,X}(D)\} \cup \{\rho_{G,X}(X(\beta)) \mid \beta \in \Phi_+, X(-\beta) = D\}$;
- (c) other generators, different from any of the above.

We show that $c \cap \Psi^{\perp}$ is contained in c' , and recall that the latter is generated by $\rho_{G,X}(S)$ together with $\iota^*(\rho_{G,X}(d(c)))$. An element $x \in c \cap \Psi^{\perp}$ is a linear combination with non-negative coefficients of the above generators, and we may assume that the elements of (a) do not contribute. This indeed implies the general case, since $\rho_{G,X}(S) \subseteq c'$.

Also, we may suppose that any generator z involved in the linear combination giving x satisfies $\iota^*(z) \neq 0$. Indeed, otherwise we may suppress it using the fact that $x = \iota^*(x)$. Hence, all generators in the linear combination of x are not of the form $\rho_{G,X}(X(\beta))$ for $\beta \in \Psi$.

It remains the generators $\rho_{G,X}(D)$ where $D \in d(c)$, and generators of (c) of the form $\rho_{G,X}(X(-\alpha))$ for some $\alpha \in \Psi$. In the second case $X(-\alpha)$ is a color, because it cannot be equal to $X(\beta)$ for any $\beta \in \Psi$. Being not in $d(c)$, each such $X(-\alpha)$ admits a positive root β satisfying $X(-\beta) = X(-\alpha)$ and D_{β} not a generator of c . This implies that β is non-positive on c , and the only chance for x to be in β^{\perp} is that such a generator $X(-\beta) = X(-\alpha)$ doesn't occur.

As a consequence, x is a linear combination of the elements $\rho_{G,X}(D)$ with $D \in d(c)$, and we easily conclude that $x \in c'$ using again $\iota^*(x) = x$.

Finally, let $x \in c'$, and let us show that $x \in c \cap \Psi^\perp$. As before, we ignore the generators of c' lying in Ψ^\perp , and we assume that x is a linear combination with non-negative coefficients of $\iota^*(d(c))$. In other words:

$$x = \sum_{\alpha \in \Psi, X(-\alpha) \in d(c)} a_\alpha \iota^*(\rho_{G,X}(X(-\alpha)))$$

with $a_\alpha \geq 0$. Consider a summand $a_\alpha \iota^*(\rho_{G,X}(X(-\alpha)))$. For each positive root $\beta \neq \alpha$ such that $X(-\beta) = X(-\alpha)$, Lemma 8.6 implies that $\gamma = \beta - \alpha$ and $-\gamma$ are also roots in Φ , and that $X(-\alpha) = X(-\beta)$, $X(\alpha) = X(-\gamma)$, $X(\gamma) = X(\beta)$ are three distinct prime divisors. Then, we take the sum

$$(8.9) \quad y = \sum_{\alpha \in \Psi, X(-\alpha) \in d(c)} a_\alpha y_\alpha$$

where

$$y_\alpha = \rho_{G,X}(X(-\alpha)) + \sum_{\substack{\beta \in \Phi_+, \\ X(-\beta) = X(-\alpha)}} \rho_{G,X}(X(\beta)).$$

We claim that all simple roots in Ψ are zero on this element, hence $\iota^*(y) = y$ and we immediately conclude that $y = x$. On the other hand, y is in c thanks to the definition of the set $d(c)$, therefore $x \in c \cap \Psi^\perp$.

Let us prove the claim. Let $\gamma \in \Psi$, and pick a y_α . If $\gamma = \alpha$, then it is easy to check using the last assertion of Proposition 8.8 that y_α is the sum of $\rho_{G,X}(X(-\alpha))$ and $\rho_{G,X}(X(\alpha))$, plus other terms where α is zero. It follows $\langle y_\alpha, \gamma \rangle = 0$.

If $\gamma \neq \alpha$, then $\langle \rho_{G,X}(X(-\alpha)), \gamma \rangle = 0$ thanks to Proposition 8.8. Moreover, in this case γ does not appear as a β in the sum expressing y_α , because we know that $X(-\alpha) \neq X(-\gamma)$. Also, if $X(\pm\gamma)$ is different from $\rho_{G,X}(X(\beta))$ for all $\beta \in \Phi_+$ such that $X(-\beta) = X(-\alpha)$, then again $\langle y_\alpha, \gamma \rangle = 0$.

Therefore we may suppose that γ is different from all the β appearing in the expression of y_α , but some of them, say $\beta_{i,\gamma}$ for $i = 1, \dots, k$, satisfy $X(\beta_{i,\gamma}) = X(\epsilon_{i,\gamma}\gamma)$ where $\epsilon_{i,\gamma} = 1$ or -1 . In this case Lemma 8.6 implies that $\beta_{i,\gamma} - \epsilon_{i,\gamma}\gamma$ also appears in the sum, with $X(\beta_{i,\gamma} - \epsilon_{i,\gamma}\gamma) =$

$X(-\epsilon_{i,\gamma}\gamma)$. We obtain:

$$\begin{aligned}
y_\alpha &= \rho_{G,X}(X(-\alpha)) + \sum_{i=1}^k (\rho_{G,X}(X(\beta_{i,\gamma})) + \rho_{G,X}(X(\beta_{i,\gamma} - \epsilon_{i,\gamma}\gamma))) \\
&\quad + \sum_{\substack{\beta \in \Phi_+, X(\beta) \neq X(\pm\gamma) \\ X(-\beta) = X(-\alpha)}} \rho_{G,X}(X(\beta)) \\
&= \rho_{G,X}(X(-\alpha)) + \sum_{i=1}^k (\rho_{G,X}(X(\gamma)) + \rho_{G,X}(X(-\gamma))) + \sum_{\substack{\beta \in \Phi_+, X(\beta) \neq X(\pm\gamma) \\ X(-\beta) = X(-\alpha)}} \rho_{G,X}(X(\beta)).
\end{aligned}$$

From this expression it is evident that $\langle y_\alpha, \gamma \rangle = 0$, and the proof of (3) is complete.

To finish the proof of the theorem, we must check that all colored cones of $\mathcal{F}_A(X)$ appear as $(c \cap \Psi^\perp, d(c))$ for some $c \in \mathcal{F}_G(X)$. For this, it is enough to notice that for each A -orbit Z there is a G -orbit Y such that $AY = Z$. \square

Corollary 8.12. *The A -variety is horospherical, i.e. $\Sigma_A(X) = \emptyset$.*

Proof. There exists a smooth complete toroidal A -variety Y equipped with a surjective birational A -equivariant morphism $Y \rightarrow X$ (it is enough to choose an A -equivariant resolution of singularities of the variety given in [Kn91, Lemma 5.2], where X'' in the proof of *loc.cit.* is our X).

Then $\Sigma_A(Y) = \Sigma_A(X)$, and Y is also a complete G -regular embedding. Applying Theorem 8.11 to Y , it follows that $\text{supp } \mathcal{F}_A(Y)$ is a vector space, and it is equal to $V_A(Y)$ because Y is toroidal and complete. We conclude that $\Sigma_A(Y) = \emptyset$. \square

Remark 8.13. With a slightly more involved proof, one can derive the above corollary directly from Proposition 8.8 and avoid using Theorem 8.11.

Remark 8.14. It is easy to check that $d(c) = \emptyset$ if and only if $c \cap \Psi^\perp$ is a face of c .

Example 8.15. Let us compute the colored fan of $X = \mathbb{P}^2$, as in Example 8.10 with $n = 2$. Choose $\mathcal{E} = \{E_3\}$ where $E_i = \{x_i = 0\}$ and x_1, x_2 and x_3 are homogeneous coordinates on \mathbb{P}^2 . Then $A = \text{Aut}^\circ(\mathbb{P}^2, \mathcal{D})$ is isomorphic to $\text{SL}(2)$, and we choose the Borel subgroup of A stabilizing the point $[1, 0, 0]$. The lattice $\Lambda_G(\mathbb{P}^2)$ is then the root lattice of $\text{PGL}(3)$, and we have $X_1 = X(\alpha_1) = X(\alpha_1 + \alpha_2)$, $X_2 = X(\alpha_2) = X(-\alpha_1)$ and $X_3 = X(-\alpha_1 - \alpha_2) = X(-\alpha_2)$, where α_1 and α_2 are the simple roots of $\text{PGL}(3)$. The lattice $\Lambda_A(\mathbb{P}^2)$ is $\rho_{G, \mathbb{P}^2}(X_1)^\perp = \mathbb{Z}\alpha_2$, which is the weight lattice of $\text{SL}(2)$, and \mathbb{P}^2 has only one A -color, namely X_2 . The maximal colored cones of $\mathcal{F}_A(\mathbb{P}^2)$ are $(\mathbb{Q}_{\geq 0}\rho_{G, \mathbb{P}^2}(X_3), \emptyset)$ and $(-\mathbb{Q}_{\geq 0}\rho_{G, \mathbb{P}^2}(X_3), \{X_2\})$.

9. SEMISIMPLE CASE

In this section we assume that G is a semisimple group, i.e. $C = \{e\}$. In this setting the functionals associated to the colors of X generate $N_G(X)$ as a vector space. Indeed, if $\lambda \in \Lambda_G(X)$ is in $\rho_{G,X}(\Delta_G(X))^\perp$, then a rational function $f \in \mathbb{C}(G/H)_\lambda^{(B)}$ is regular on G/H and nowhere zero. It can be then lifted to a nowhere-vanishing function $F \in \mathbb{C}[G]$, which is then constant since G has no non-trivial character (see [KKV89, Proposition 1.2]). We conclude that $\lambda = 0$, and the claim follows.

This essentially implies the following main result of this section.

Theorem 9.1. *If G is semisimple and \mathcal{D} is any subset of $\partial_G X$, then $\text{Aut}^\circ(X, \mathcal{D} \cup (\partial_G X)^\ell)$ is a Levi subgroup of $\text{Aut}^\circ(X, \mathcal{D})$.*

The proof is at the end of this section. The theorem implies that if G is semisimple then §7 is enough to describe a Levi subgroup of $\text{Aut}^\circ(X, \mathcal{D})$ and its action on X , without any restriction on \mathcal{D} .

Recall from §5 the restriction map

$$\kappa_{x'} : (\ker \psi_*)^\circ \rightarrow \text{Aut}^\circ(X_{x'})$$

where x' lies on the open G -orbit of X' , and $X_{x'} = \psi^{-1}(x')$.

Lemma 9.2. *For all x' in the open B -orbit of X' , the image of $\kappa_{x'}$ in $\text{Aut}^\circ(X_{x'})$ is very solvable (i.e. contained in a Borel subgroup).*

Proof. To simplify notations we assume that $x' = x'_0$. Let $\mathcal{E}' \subseteq \partial_S X_{x'_0}$ be the following subset:

$$\mathcal{E}' = \{E \cap X_{x'_0} \mid E \in \mathcal{E}\},$$

and define $\mathcal{D}' = \partial_S X_{x'_0} \setminus \mathcal{E}'$. Let us also denote by $K_{x'_0}$ the image of $\kappa_{x'_0}$: it is obviously a subgroup of $\text{Aut}^\circ(X_{x'_0}, \mathcal{D}')$. On the other hand $K_{x'_0}$ contains the maximal torus S of $\text{Aut}^\circ(X_{x'_0})$, hence we only have to compute the root subgroup it contains. Thanks to Lemma 6.4 and Corollary 5.3, they are the root spaces $U_\alpha \subset \text{Aut}^\circ(X_{x'_0})$ for α varying in the set

$$R = \{\gamma|_S \mid 0 \neq \gamma \in \Lambda_G(G/H), X(\gamma) \text{ exists and } X(\gamma) \in \mathcal{E}^\ell\}.$$

From Lemma 6.5, we obtain that R doesn't contain the opposite of any of its elements, therefore $K_{x'_0}$ is very solvable. \square

Proof of Theorem 9.1. First, observe that $(\ker \psi_*)^\circ$ is solvable. This stems from Lemma 9.2, and the obvious observation that

$$(9.1) \quad \bigcap_{x' \text{ in the open } B\text{-orbit of } X'} \ker(\kappa_{x'}) = \{\text{id}_X\}.$$

Consider now the variety X under the action of $A = \text{Aut}^\circ(X, \mathcal{D} \cup (\partial_G X)^\ell)$. Thanks to Theorem 7.8, the group A is semisimple (because here G is semisimple) and under its action X is a G -regular embedding with boundary $\mathcal{D} \cup (\partial_G X)^\ell$. Corollary 7.19 implies $(\partial_A X)^{n\ell} = \mathcal{D}^{n\ell} \subseteq \mathcal{D}$, and we deduce that $\text{Aut}^\circ(X, \mathcal{D}) \subseteq \text{Aut}^\circ(X, (\partial_A X)^{n\ell})$.

Then we may apply Proposition 5.2 with G replaced by the universal cover of A : the theorem follows. \square

Remark 9.3. Let X and G be as in Example 7.9. Then the full automorphism group of X is non-reductive. Indeed, it must fix the point $p \in \mathbb{P}^{n+1}$, and one concludes easily that $\text{Aut}^\circ(X)$ is the corresponding maximal proper parabolic subgroup of $\text{PGL}(n+2) \times \text{PGL}(n+1)$. The unipotent radical $\text{Aut}^\circ(X)^u$ can be studied restricting its elements to the generic fiber $X_{x'_0}$; however, the example shows that for any given fiber the restriction may be non-injective, therefore a global analysis of these restrictions is needed. This goes beyond the scope of the present work.

10. G -STABLE PRIME DIVISORS ON THE LINEAR PART OF THE VALUATION CONE

In this section $G = G' \times C$ is neither abelian nor semisimple. For simplicity, and thanks to §8, we may assume that G' acts non-trivially on X . The variety X' is then not a single point. Recall that S acts on X naturally by G -equivariant automorphisms preserving the fibers of ψ , so we can consider S as a subgroup of $\text{Aut}^\circ(X, \partial_G X) \cap (\ker \psi_*)^\circ$.

We study the automorphism group $\text{Aut}^\circ(X, \mathcal{D})$, where $\mathcal{D} \subseteq \partial_G X$ satisfies $\mathcal{D} \supseteq (\partial_G X)^{n\ell}$. Denote as usual $\mathcal{E} = \partial_G X \setminus \mathcal{D} \subseteq (\partial_G X)^\ell$, and recall that all elements $D \in (\partial_G X)^\ell$ intersect $X_{x'_0}$ in an S -stable prime divisor.

Proposition 10.1. *Let x' in the open G -orbit of X' , and $L = L(X, \mathcal{D})$ be a Levi subgroup of $(\text{Aut}^\circ(X, \mathcal{D}) \cap \ker(\psi_*))^\circ$ containing S . Then $L_{x'} = \kappa_{x'}(L)$ is isomorphic to L , and the group*

$$(\theta_{G,X}(G), \theta_{G,X}(G)) \times L(X, \mathcal{D})$$

is locally isomorphic to a Levi subgroup of $\text{Aut}^\circ(X, \mathcal{D})$.

Proof. Thanks to formula (9.1) and the fact that x' is generic in X' , we know that the map $L \rightarrow L_{x'}$ has unipotent kernel, therefore is an isomorphism. The rest follows from Proposition 5.2. \square

Definition 10.2. We define the following group:

$$A = A(X, \mathcal{D}) = (\theta_{G,X}(G), \theta_{G,X}(G)) \times L(X, \mathcal{D}),$$

where $L(X, \mathcal{D})$ is defined as in Proposition 10.1.

We describe now the reductive group $L_{x'_0}$ in terms of the root subspaces it contains with respect to its maximal torus S .

Definition 10.3. We define

$$R = R(X, \mathcal{D}) = \{\gamma|_S \mid 0 \neq \gamma \in \Lambda_G(G/H), X(\gamma) \text{ exists and } X(\gamma) \in \mathcal{E}\},$$

and we denote by $\Phi = \Phi(X, \mathcal{D})$ the maximal subset of R such that $-\alpha \in R$ for every $\alpha \in R$.

Proposition 10.4. *The set $\Phi(X, \mathcal{D})$ is a subset of $\Phi(X_{x'_0}, \mathcal{D}')$, where $\mathcal{D}' = \{D \cap X_{x'_0} \mid D \in \mathcal{D}^\ell\}$. Moreover, $L_{x'_0} \subseteq \text{Aut}^\circ(X_{x'_0})$ is generated by S together with all subgroups U_α such that $\alpha \in \Phi(X, \mathcal{D})$.*

Proof. For the first assertion, it is enough for any $\alpha = \gamma|_S \in \Phi(X, \mathcal{D})$ to restrict the function $f \in H^0(X, \mathcal{O}_X(X(\gamma)))_{\gamma}^{(B)}$ to $X_{x'_0}$. Since S is a maximal torus of $\text{Aut}^\circ(X_{x'_0})$, the second assertion follows from Lemma 6.4 and Corollary 5.3. \square

This provides a complete description of the group A . It remains now to describe the fan associated to X as an A -variety.

Let $0 \neq \gamma \in \Lambda(G/H)$ be such that $\gamma|_S = \alpha \in \Phi$, and choose $f_\gamma \in H^0(X, \mathcal{O}_X(X(\gamma)))_{\gamma}^{(B)}$ such that $f_\gamma(x_0) = 1$. Then $\rho_{G,X}(X(\gamma)) \in V_G^\ell(G/H)$ can be considered as an element of $\text{Hom}_{\mathbb{Z}}(\mathcal{X}(S), \mathbb{Z})$, and therefore it is canonically associated with a 1-PSG $\mu_\gamma: \mathbb{C}^* \rightarrow S$. The torus S acts on X through the identification with a subtorus of $T_{G,X}$, as we have seen in §5; in this way μ_γ induces a tangent vector field $\delta_\gamma \in H^0(X, \mathcal{T}_X)$ on X .

Lemma 10.5. *The product $\xi_\gamma = f_\gamma \delta_\gamma$ is a well-defined tangent vector field of X , and it is sent to $H^0(X, \mathcal{O}_X(X(\gamma)))$ via the surjective map of (4.1). Its restriction to $X_{x'_0}$ is a tangent vector field and is a generator of the Lie algebra of $U_\alpha \subset \text{Aut}^\circ(X_{x'_0})$. Moreover, the 1-PSG of $\text{Aut}^\circ(\mathcal{Z}_{G,X})$ induced by ξ_γ is expressed in local coordinates by the formulae of [Oda88, Proposition 3.14].*

Proof. The rational function f_γ has its only pole in $X(\gamma)$, which means that we only have to check the first assertion on points of $X(\gamma)$. On $\mathcal{Z}_{G,X} \cap X(\gamma)$ it can be checked easily using the fact that $\mathcal{Z}_{G,X}$ is a toric $T_{G,X}$ -variety, and expressing ξ_γ in local coordinates. This also implies that ξ_γ is a well-defined vector field on $E \cap X_0$, thanks to the $P_{G,X}^u$ -invariance of both f_γ and δ_γ . Then the locus where ξ_γ might not be a well-defined vector field has codimension at least 2, which implies the first statement.

Since S acts on X stabilizing both $\mathcal{Z}_{G,X}$ and $X_{x'_0}$, we deduce that ξ_γ can be restricted to a vector field on both these varieties. The rest follows easily by expressing ξ_γ on $\mathcal{Z}_{G,X}$ explicitly in local coordinates. \square

Definition 10.6. We choose a Borel subgroup B_A of A such that $\theta_{A,X}(B_A) \cap \theta_{G,X}(G) = \theta_{G,X}(B)$ and such that $B_A \cap L$ is a Borel subgroup of L . Let us also denote by $\Psi = \Psi(X, \mathcal{D}) \subset \Phi(X, \mathcal{D})$ the set of simple roots and by $\Phi_+ = \Phi_+(X, \mathcal{D}) \subset \Phi(X, \mathcal{D})$ the set of positive roots associated to the Borel subgroup $B_{L_{x'_0}} = \kappa_{x'_0}(B_A \cap L)$ of $L_{x'_0}$. Finally, let

$$r: \Lambda_G(X) \rightarrow \mathcal{X}(S) = \Lambda_S(X_{x'_0})$$

be the restriction of characters of $\Lambda_G(X)$ to S (see §5).

We may apply Proposition 8.8 and Theorem 8.11 to the toric S -variety $X_{x'_0}$ and the sets of roots Φ and Ψ . We obtain a description of $X_{x'_0}$ as an $L_{x'_0}$ -variety, and in particular the lattice

$$\Lambda_{L_{x'_0}}(X_{x'_0}) \subseteq \Lambda_S(X_{x'_0}),$$

together with the projection

$$N_S(X_{x'_0}) \rightarrow N_{L_{x'_0}}(X_{x'_0}).$$

Proposition 10.7. *The restriction of weights from $\theta_{A,X}(B_A)$ to $\theta_{G,X}(B)$ induces an isomorphism*

$$\Lambda_A(X) \cong r^{-1}(\Lambda_{L_{x'_0}}(X_{x'_0})) \subseteq \Lambda_G(X).$$

We denote the corresponding surjective map by

$$s: N_G(X) \rightarrow N_A(X).$$

The set of colors of X as a spherical A -variety is the following disjoint union:

$$\Delta_A(X) = \Delta_G(X) \cup \{E \in \mathcal{E} \mid E \cap X_{x'_0} \text{ is a color of the spherical } L_{x'_0}\text{-variety } X_{x'_0}\},$$

and for each $E \in \Delta_A(X)$, we have

$$\rho_{A,X}(E) = s(\rho_{G,X}(E)).$$

Proof. A B_A -eigenvector in $\mathbb{C}(X)$ is a fortiori a B -eigenvector, thanks to the choice of B_A . This induces an inclusion $\Lambda_A(X) \subseteq \Lambda(X)$.

Moreover, a B -eigenvector $f \in \mathbb{C}(X)$ is also a B_A -eigenvector if and only if its restriction $f|_{X_{x'_0}}$ is a $B_{L_{x'_0}}$ -eigenvector, thanks to the structure of A as described in Proposition 10.1. This proves the first assertion.

Secondly, a color of X as an A -variety maps either dominantly onto X' , or not. In the first case, its intersection with the (generic) fiber $X_{x'_0}$ is $B_{L_{x'_0}}$ -stable but not $L_{x'_0}$ -stable (otherwise it would have been A -stable).

In the second case, it maps onto a G -color of X' , i.e. it is a color of X with respect to the G action. The second assertion follows. \square

Let c be a cone of the fan $\mathcal{F}(X)$. Then c is generated as a convex cone by a set of 1-dimensional faces $F(c)$. We denote by c^ℓ the intersection $c \cap V_G^\ell(X)$, by $F^\ell(c)$ the 1-dimensional faces of $F(c)$ generating c^ℓ , and $F^{n\ell}(c) = F(c) \setminus F^\ell(c)$.

Since c^ℓ is a cone of the toric S -variety $X_{x'_0}$, it corresponds to an S -orbit Y on $X_{x'_0}$. As in the proof of Theorem 8.11, the corresponding $L_{x'_0}$ -orbit $L_{x'_0}Y$ on $X_{x'_0}$ has colored cone $(c^\ell \cap \Psi^\perp, d(c^\ell))$, where the orthogonal Ψ^\perp is taken inside $V_G^\ell(G/H)$, and $d(c^\ell)$ is a set of $L_{x'_0}$ -colors of $X_{x'_0}$.

Definition 10.8. For any $c \in \mathcal{F}(X)$, we define a colored cone $(c_A(c), d_A(c))$, where $c_A(c) \subset N_A(X)$ and $d_A(c) \subseteq \Delta_A(X)$, as follows. The cone $c_A(c)$ is the convex cone in $N_A(X)$ generated by $s(F^{n\ell}(c))$ and $s(c^\ell \cap \Psi^\perp)$. The set $d_A(c)$ is the set of colors $E \in \Delta_A(X)$ such that $E \notin \Delta_G(X)$, and $E \cap X_{x'_0} \in d(c^\ell)$.

Theorem 10.9. *The colored fan $\mathcal{F}_A(X)$ as an A -variety is*

$$\mathcal{F}_A(X) = \{(c_A(c), d_A(c)) \mid c \in \mathcal{F}_G(X)\}.$$

Proof. Let Y be a G -orbit of X , with associated cone $c = c_{X,Y}$. We claim that the colored cone associated to the A -orbit AY is $(c_A(c), d_A(c))$: arguing as in the proof of Theorem 8.11, this is enough to show the theorem.

To prove the claim, first we show that the set d' of A -colors containing AY is equal to $d_A(c)$. Since X is toroidal, no G -color contains Y , nor AY . Therefore any A -color E in d' is indeed a G -stable prime divisor whose functional lies in $V_G^\ell(X)$. It intersects $X_{x'_0}$ in an $L_{x'_0}$ -color of $X_{x'_0}$, by Proposition 10.7, and we only have to show that $E \cap X_{x'_0}$ is in $d(c^\ell)$.

We check this fact using the definition of $d(c^\ell)$. Take a positive root $\beta \in \Phi_+$ of $X_{x'_0}$, the prime divisors $X_{x'_0}(\beta)$, $X_{x'_0}(-\beta)$ of $X_{x'_0}$ as in Definition 8.1, and suppose that $X_{x'_0}(-\beta) = E \cap X_{x'_0}$, so $\rho_{S, X_{x'_0}}(X_{x'_0}(-\beta))$ lies on a 1-codimensional face of c^ℓ . We have to show that $\rho_{S, X_{x'_0}}(X_{x'_0}(\beta))$ also lies on a 1-codimensional face of c^ℓ , in other words that $X_{x'_0}(\beta)$ contains the S -orbit of $X_{x'_0}$ associated c^ℓ .

Now $E = E_1$ and some other element $E_2 \in \mathcal{E}$ satisfy $E_1 \cap X_{x'_0} = X_{x'_0}(-\beta)$, $E_2 \cap X_{x'_0} = X_{x'_0}(\beta)$, and $-\beta$ and β are the restrictions to S of resp. $\gamma_1, \gamma_2 \in \Lambda_G(X)$, such that $X(\gamma_i) = E_i$ for $i = 1, 2$. Suppose that E_2 doesn't contain Y . Then we consider $\mathcal{Z}_{G,X}$: intersecting it with E_1 , E_2 and Y two $T_{G,X}$ -stable prime divisors and a $T_{G,X}$ -orbit, such that $E_1 \cap \mathcal{Z}_{G,X} \supseteq Y \cap \mathcal{Z}_{G,X}$ and $E_2 \cap \mathcal{Z}_{G,X} \not\supseteq Y \cap \mathcal{Z}_{G,X}$.

At this point we follow the same approach of the proof of Theorem 8.11, statement (2), applied to the toric variety $\mathcal{Z}_{G,X}$ and the automorphisms induced by the tangent vector field ξ_{γ_1} (as defined in Lemma 10.5). This yields the formula (8.6) for ξ_{γ_1} , which shows that $E_1 \cap \mathcal{Z}_{G,X}$ doesn't contain $AY \cap \mathcal{Z}_{G,X}$: a contradiction. As a consequence $E_2 \supseteq Y$, so $X_{x'_0}(\beta)$ contains the S -orbit of $X_{x'_0}$ associated c^ℓ . This concludes the proof of the inclusion $d' \subseteq d_A(c)$.

Let now $D \in d_A(c)$. Then, by Theorem 8.11, the intersection $D \cap X_{x'_0}$ contains the $L_{x'_0}$ -orbit of $X_{x'_0}$ corresponding to $(c^\ell \cap \Psi^\perp, d(c^\ell))$. Let y be a point on this orbit: then D contains \overline{Ay} .

On the other hand, from the proof of Theorem 8.11, we see that $\overline{L_{x'_0}y}$ contains the S -orbit of $X_{x'_0}$ corresponding to $c^\ell \subset N_S(X_{x'_0})$. It follows that \overline{Ay} contains the G -orbit of X associated to $c^\ell \subset N_G(X)$, and thus also the G -orbit Y associated to $c \subset N_G(X)$. Being A -stable, \overline{Ay} must then contain AY too, and since D is closed, we obtain $D \supseteq AY$. I.e., D is in d' .

We now prove that the convex cone c' associated to AY is $c_A(c)$. First observe that Y and AY are contained in the same elements of $(\partial_G X)^{n\ell}$, since L stabilizes all fibers of ψ . Therefore

c' is generated by $s(F^{n\ell}(c))$ and its intersection with $s(V_G^\ell(X))$. It remains to prove that $c' \cap s(V_G^\ell(X)) = s(c^\ell \cap \Psi^\perp)$.

The cone $c' \cap s(V_G^\ell(X))$ is generated by $\rho_{A,X}(E)$ where $E \in (\partial_G X)^\ell$ is:

- (1) an A -color of X containing AY , i.e. $E \in d_A(c)$, or
- (2) an A -stable prime divisor containing AY .

On the other hand the generators of $s(c^\ell \cap \Psi^\perp)$ are the elements $\rho_{A,X}(E)$ where $E \in (\partial_G X)^\ell$ is:

- (1') an A -color such that $E \cap X_{x'_0}$ is a color containing the $L_{x'_0}$ -orbit Z of $X_{x'_0}$ associated to $(c^\ell \cap \Psi^\perp, d(c^\ell))$, or
- (2') an A -stable prime divisor such that $E \cap X_{x'_0}$ is a $L_{x'_0}$ -stable prime divisor containing Z .

Thanks to the first part of the proof, the prime divisors E of type (1) and of type (1') are the same.

If E is of type (2') then it contains AZ , whose closure in turn contains AY . Therefore E is of type (2). Let now E be of type (2). Then $E \cap \mathcal{Z}_{G,X}$ is an L -stable (and $T_{G,X}$ -stable) prime divisor of $\mathcal{Z}_{G,X}$ containing $Y \cap \mathcal{Z}_{G,X}$, which is the $T_{G,X}$ -orbit of $\mathcal{Z}_{G,X}$ associated to c , and $\rho_{T_{G,X}, \mathcal{Z}_{G,X}}(E \cap \mathcal{Z}_{G,X})$ lies on $V_G^\ell(G/H)$. Hence $E \cap X_{x'_0}$ is an $L_{x'_0}$ -stable prime divisor of $X_{x'_0}$ containing the S -orbit of $X_{x'_0}$ associated to c^ℓ . Thanks to the proof of Theorem 8.11, we deduce that $E \cap X_{x'_0}$ contains Z , i.e. E is of type (2'). \square

Corollary 10.10. $\Sigma_A(X) = \Sigma_G(X)$.

Proof. The proof is similar to the proof of Corollary 8.12. \square

REFERENCES

- [AG10] I. Arzhantsev, S.A. Gaifullin: *Homogeneous toric varieties*, Journal of Lie Theory **20** (2010), no. 2, 283–293.
- [BB96] F. Bien, M. Brion: *Automorphisms and local rigidity of regular varieties*, Compositio Math. **104** (1996), no. 1, 1–26.
- [BDP90] E. Bifet, C. De Concini, C. Procesi: *Cohomology of regular embeddings*, Adv. in Math. **82** (1990), 1–34.
- [Br89] M. Brion: *Groupe de Picard et nombres caractéristiques des variétés sphériques*. Duke Math. J. **58** (1989), no. 2, 397–424.
- [Br90] M. Brion: *Vers une généralisation des espaces symétriques*, J. Algebra **134** (1990), no. 1, 115–143.
- [Br97] M. Brion: *Variétés sphériques*, <http://www-fourier.ujf-grenoble.fr/~mbrion/spheriques.ps>
- [Br07] M. Brion: *The total coordinate ring of a wonderful variety*, J. Algebra **313** (2007), no. 1, 61–99.
- [BP87] Brion, F. Pauer. Valuations des espaces homogènes sphériques. *Commentarii Mathematici Helvetici* **62** (1987), no. 2, 265–285.
- [De70] M. Demazure, *Sous-groupes algébriques de rang maximum du groupe de Cremona*, Ann. Sci. École Norm. Sup. **3**, (1970), no. 4, 507–588.
- [De77] M. Demazure, *Automorphismes et déformations des variétés de Borel*, Invent. Math. **39** (1977), no. 2, 179–186.

- [Gi89] V. Ginzburg: *Admissible modules on a symmetric space*, Astérisque **173–174** (1989), 199–255.
- [Kn91] F. Knop: *The Luna-Vust theory of spherical embeddings*, Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989), 225–249, Manoj Prakashan, Madras, 1991.
- [Kn94] F. Knop: *The asymptotic behaviour of invariant collective motion*, Invent. Math. **114**, 1994, 309–328.
- [Kn96] F. Knop: *Automorphisms, root systems, and compactifications of homogeneous varieties*, J. Amer. Math. Soc. **9** (1996), no. 1, 153–174.
- [KKLV89] F. Knop, H. Kraft, D. Luna, T. Vust: *Local properties of algebraic group actions*, in *Algebraische Transformationsgruppen und Invariantentheorie* (H. Kraft, P. Slodowy, T. Springer eds.) DMV-Seminar 13, Birkhäuser Verlag (Basel-Boston) (1989) 63–76.
- [KKV89] F. Knop, H. Kraft, T. Vust: *The Picard group of a G -variety*, in *Algebraische Transformationsgruppen und Invariantentheorie* (H. Kraft, P. Slodowy, T. Springer eds.) DMV-Seminar 13, Birkhäuser Verlag (Basel-Boston) (1989) 77–87.
- [Lu01] D. Luna: *Variétés sphériques de type A* , Inst. Hautes Études Sci. Publ. Math. **94** (2001), 161–226.
- [Oda88] T. Oda: *Convex Bodies and Algebraic Geometry: An Introduction to Toric Varieties*, Springer-Verlag, New York-Berlin-Heidelberg, 1988.
- [Pe09] G. Pezzini: *Automorphisms of wonderful varieties*, Transform. Groups **14** (2009), no. 3, 677–694.
- [Wa96] B. Wasserman: *Wonderful varieties of rank two*, Transform. Groups **1** (1996), no. 4, 375–403.

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